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THE DIAGONAL ENTRIES IN THE FORMULA 'QUASITRIANGULAR – COMPACT = TRIANGULAR', AND RESTRICTIONS OF QUASITRIANGULARITY

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ABSTRACT. A (bounded linear) Hilbert space operator T is called *quasitriangular* if there exists an increasing sequence $\{P_n\}_{n=0}^{\infty}$ of finite-rank orthogonal projections, converging strongly to 1, such that $\|(1-P_n)TP_n\| \to 0$ $(n \to \infty)$. This definition, due to P. R. Halmos, plays a very important role in operator theory. The core of this article is a concrete answer to the following problem: Suppose T is a quasitriangular operator and $\Gamma = \{\lambda_j\}_{j=1}^{\infty}$ is a sequence of complex numbers. Find necessary and sufficient conditions for the existence of a compact operator K (of arbitrarily small norm) so that $T-K$ is triangular with respect to some orthonormal basis, and the sequence of diagonal entries of $T-K$ coincides with Γ. For instance, if no restrictions are put on the norm of K, then T and Γ must be related as follows: (a) if λ_0 is a limit point of Γ and $\lambda_0 - T$ is semi-Fredholm, then $\operatorname{ind}(\lambda_0 - T) > 0$; and (b) if Ω is an open set intersecting the Weyl spectrum of T, whose boundary does not intersect this set, then $\{j: \lambda_j \in \Omega\}$ is a denumerable set of indices.

Particularly important is the case when $\Gamma = \{0,0,0,\dots\}$. The following are equivalent for an operator T: (1) there is an integral sequence $\{P_n\}_{n=0}^{\infty}$ of orthogonal projections, with $\operatorname{rank} P_n = n$ for all n, converging strongly to 1, such that $\|(1-P_n)TP_{n+1}\| \to 0$ $(n \to \infty)$; (2) from some compact K, $T-K$ is triangular, with diagonal entries equal to 0; (3) T is quasitriangular, and the Weyl spectrum of T is connected and contains the origin. The family $(\mathrm{StrQT})_{-1}$ of all operators satisfying (1) (and hence (2) and (3)) is a (norm) closed subset of the algebra of all operators; moreover, $(\mathrm{StrQT})_{-1}$ is invariant under similarity and compact perturbations and behaves in many senses as an analog of Halmos's class of quasitriangular operators, or an analog of the class of extended quasitriangular operators $(\mathrm{StrQT})_1$, introduced by the author in a previous article.

If $\{P_n\}_{n=0}^{\infty}$ is as in (1), but condition $\|(1-P_n)TP_{n+1}\| \to 0$ $(n \to \infty)$ is replaced by (1′) $\|(1-P_{n_k})TP_{n_k+1}\| \to 0$ $(k \to \infty)$ for some subsequence $\{n_k\}_{k=1}^{\infty}$, then (1′) is equivalent to (3′), T is quasitriangular, and its Weyl spectrum contains the origin. The family $(\mathrm{QT})_{-1}$ of all operators satisfying (1′) (and hence (3′)) is also a closed subset, invariant under similarity and compact perturbations, and provides a different analog to Halmos's class of quasitriangular operators. Both classes have "m-versions" ($(\mathrm{StrQT})_{-m}$ and, respectively, $(\mathrm{QT})_{-m}$, $m = 1,2,3,\dots$) with similar properties. ($(\mathrm{StrQT})_{-m}$ is the class naturally associated with triangular operators A such that the main diagonal and the first $(m-1)$ superdiagonals are identically zero, etc.)

The article also includes some applications of the main result to certain nest algebras "generated by orthonormal bases."

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1. Introduction. Suppose that an operator A, acting on a complex separable infinite dimensional Hilbert space $\mathscr{H}$, admits a triangular matrix $(a_{ij})_{i,j=1}^{\infty}$ with respect to an orthonormal basis of $\mathscr{H}$ (that is, $a_{ij}=0$ for all $i>j$). The sequence $d(A)=\{a_{jj}\}_{j=1}^{\infty}$ of all diagonal entries of A necessarily satisfies certain inclusion relations with respect to the spectrum $\sigma(A)$ of A and its different parts. In order to make this clearer, we shall need some notation; $\sigma_l(T)$ ($\sigma_r(T)$) is the left (right, respectively) spectrum of the operator T. If $\mathscr{L}(\mathscr{H})$ denotes the algebra of all (bounded linear) operators acting on $\mathscr{H}$, $\mathscr{K}(\mathscr{H})$ denotes the ideal of all compact operators, and $\mathscr{A}(\mathscr{H})=\mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ is the quotient Calkin algebra, then

$$\sigma_e(T) = \text{the spectrum of the canonical image, } \tilde{T}, \text{ of } T \text{ in } \mathscr{A}(\mathscr{H})$$

is the essential spectrum of T; $\sigma_e(T)$ is the union of the left essential spectrum, $\sigma_{le}(T)$, and the right essential spectrum, $\sigma_{re}(T)$, of T. The intersection, $\sigma_{lre}(T)=\sigma_{le}(T)\cap\sigma_{re}(T)$, of these two sets is called the Wolf spectrum of T; its complement, $\rho_{s\text{-}F}(T)=\mathbf{C}\setminus\sigma_{lre}(T)$ is the semi-Fredholm domain of T.

Finally, we denote by $\sigma_p(T)$ and $\sigma_0(T)$ the point spectrum of T and, respectively, the set of all normal eigenvalues of T. ($\lambda\in\sigma(T)$ is a *normal eigenvalue* if λ is an isolated point of $\sigma(T)$ such that the corresponding Riesz subspace, $\mathscr{H}(\lambda;T)$, is finite dimensional; equivalently, λ is an isolated point of $\sigma(T)\setminus\sigma_e(T)$. The reader is referred to [**11**, Chapter 1; **19**] for the analysis of these different pieces of the spectrum, as well as for the definition and properties of semi-Fredholm operators: $\rho_{s\text{-}F}^{+}(T)=\{\lambda\in\rho_{s\text{-}F}(T)\colon \operatorname{ind}(\lambda-T)>0\}$, $\rho_{s\text{-}F}^{-}(T)=\{\lambda\in\rho_{s\text{-}F}(T)\colon \operatorname{ind}(\lambda-T)<0\}$.)

With this notation in mind, we have the following result. (In order to simplify the notation, we shall write $d(A)$ to denote both the *sequence* of all diagonal entries of the triangular operator A and the *set* $\{\lambda\in\mathbf{C}\colon \lambda=a_{jj}$ for some $j\geqslant 1\}$. No confusion will arise.)

PROPOSITION 1.1. (See [**10** and **11**, Corollary 3.40].) *If A is a triangular operator with triangular matrix $(a_{ij})_{i,j=1}^{\infty}$ (with respect to some orthonormal basis of $\mathscr{H}$) and diagonal sequence $d(A)=\{a_{jj}\}_{j=1}^{\infty}$, then*

(i) $d(A)\subset\sigma(A)=\sigma_l(A)=\sigma_{lre}(A)\cup\rho_{s\text{-}F}^{+}(A)\cup\sigma_0(A)$, *so that* $\operatorname{ind}(\lambda-A)\geqslant 0$ *for all* $\lambda\in\rho_{s\text{-}F}(A)$.

(ii) *Every nonempty clopen subset of $\sigma(A)$ intersects $d(A)$, and every component of $\sigma(A)$ intersects $d(A)^-$, the closure of $d(A)$. Furthermore, if σ is a clopen subset of $\sigma(A)$ such that $\sigma\cap\sigma_e(A)\neq\varnothing$, then* $\operatorname{card}\{j\colon a_{jj}\in\sigma\}=\aleph_0$.

(iii) *Every isolated point of $\sigma(A)$ belongs to $d(A)$. Moreover, if $\lambda\in\sigma_0(A)$, then* $\operatorname{card}\{j\colon a_{jj}=\lambda\}=\dim\mathscr{H}(\lambda;A)$.

(iv) *If $\ker(\lambda-A)^*\neq\{0\}$, then $\lambda\in d(A)$, so that $\sigma_p(A^*)$ is an at most denumerable subset of $d(A)^*=\{\bar{\lambda}\colon\lambda\in d(A)\}$.*

It is easily seen that an operator A is triangular with respect to some orthonormal basis (ONB) of its underlying space if and only if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of finite-rank orthogonal projections such that $P_n\to 1$ (strongly, as $n\to\infty$), and $P_nAP_n=AP_n$ for all $n=1,2,\ldots$.

P. R. Halmos introduced the important notion of quasitriangularity [8]: $T \in \mathcal{L}(\mathcal{H})$ is *quasitriangular* if there exists a sequence $\{P_n\}_{n=1}^{\infty}$ as above such that $\|TP_n - P_nTP_n\| \to 0$ $(n \to \infty)$. Every triangular operator is quasitriangular, and every quasitriangular operator T is the sum of a triangular operator A and a compact operator K; moreover, given $\varepsilon > 0$, K can be chosen so that $\|K\| < \varepsilon$ [9].

In [15], the author extended the notion of quasitriangularity to consider limits of 'almost-triangular' (in a certain precise sense) operators. An important piece of information for the study of this extended quasitriangularity was the partial analysis of the possible location of the diagonal entries of the triangular operators obtained from a small compact perturbation of a given quasitriangular one. The main result of this article is the (essentially) most complete possible answer to that question:

Given a quasitriangular operator T, $\varepsilon > 0$, and a sequence $\Gamma = \{\lambda_j\}_{j=1}^{\infty}$ of complex numbers such that

(i′) $\Gamma \subset \sigma_{\mathrm{lre}}(T) \cup \rho_{\text{s-F}}^{+}(T) \cup \sigma_0(T)$;

(ii′) every nonempty clopen subset σ of $\sigma(T)$ intersects Γ, and $\operatorname{card}\{j\colon \lambda_j \in \sigma\} = \aleph_0$, whenever $\sigma \cap \sigma_{\mathrm{e}}(T) \neq \varnothing$; and

(iii′) $\operatorname{card}\{j\colon \lambda_j = \lambda\} = \dim \mathcal{H}(\lambda; T)$ for each normal eigenvalue λ of T,

then there exists $K_\varepsilon \in \mathcal{K}(\mathcal{H})$, with $\|K_\varepsilon\| < \varepsilon$, such that $A_\varepsilon = T - K_\varepsilon$ is a triangular operator whose diagonal sequence $d(A)$ coincides with the sequence Γ; furthermore, K_ε can be chosen so that $A_\varepsilon | \mathcal{H}(\lambda; A_\varepsilon)$ is similar to $T \,|\, \mathcal{H}(\lambda; T)$ for all $\lambda \in \sigma_0(T)$.

Indeed, a slightly better (but more technical) result will be proven. For instance, (i′) can be relaxed to this condition:

> $\Gamma \subset \sigma_{\mathrm{lre}}(T) \cup [\text{interior } \sigma(T)] \cup \sigma_0(T)$, and $\Gamma \cap \{\lambda \in \rho_{\text{s-F}}(T)\colon \operatorname{ind}(\lambda - T) = 0\}$ only accumulates on the boundary, $\partial\sigma_{\mathrm{e}}(T)$, of the essential spectrum of T.

(See §2 below. The last section of the article is devoted to a partial analysis of the analogous problems for the cases when the orthonormal bases are totally ordered by some denumerable set of indices, not necessarily equal to the set $\mathbf{N}$ of all natural numbers.)

As a corollary of this construction (for the particular case when Γ can be chosen equal to the sequence $\{0, 0, 0, \ldots\}$), it will be shown that those results (Theorems 1 and 1^{a}) proved in [15] for two different versions of extended quasitriangularity have almost completely symmetric versions for 'restricted quasitriangularity.' It will be assumed that the reader is familiar with the results contained in this reference.

Unfortunately, in order to explain these phenomena, we must introduce some new notation. Let $\mathcal{P}(\mathcal{H})$ denote the family of all finite-rank orthogonal projections in $\mathcal{L}(\mathcal{H})$; $\mathcal{P}(\mathcal{H})$ is a directed set partially ordered by the relation $P \leqslant Q$ if $\operatorname{ran} P \subset \operatorname{ran} Q$. The modulus of quasitriangularity of $T \in \mathcal{L}(\mathcal{H})$ is the nonnegative number $q(T)$ defined by

$$q(T) = \liminf_{P \in \mathcal{P}(\mathcal{H})} \|(1 - P)TP\| = \lim_{P \in \mathcal{P}(\mathcal{H})} \left\{ \inf_{P \leqslant Q} \|(1 - Q)TQ\| \right\}.$$

The following are well-known equivalent statements for T in $\mathscr{L}(\mathscr{H})$ (see, for example, [**11**, Theorem 6.4]):

(i) T belongs to the class (QT) of all quasitriangular operators;

(ii) there exists a (not necessarily increasing) sequence $\{P_n\}_{n=1}^{\infty}$ in $\mathscr{P}(\mathscr{H})$ such that $P_n \to 1$ (strongly) and $\|(1 - P_n)TP_n\| \to 0$, as $n \to \infty$;

(iii) $q(T) = 0$;

(iv) T is the norm-limit of a sequence of triangular operators;

(v) $\operatorname{ind}(\lambda - T) \geqslant 0$ for all $\lambda \in \rho_{\text{s-F}}(T)$;

(vi) T can be decomposed as $T = A + K$, where A is triangular and K is compact

(vi)$_\varepsilon$ given $\varepsilon > 0$ there exists $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon\| < \varepsilon$, such that $A_\varepsilon = T - K_\varepsilon$ is triangular.

Furthermore, $\operatorname{dist}[R, (\mathrm{QT})] = q(R)$ for all r in $\mathscr{L}(\mathscr{H})$.

Moreover, it readily follows from the equivalence of (i) and (vi) that (i) is also equivalent to the more stringent condition:

(i′) there exists $\{P_n\}_{n=0}^{\infty} \subset \mathscr{P}(\mathscr{H})$, such that $P_n \to 1$ (strongly, as $n \to \infty$; in symbols: $P_n \uparrow 1$), $\operatorname{rank} P_n = n$ for all $n = 0, 1, 2, \ldots$, and $\|(1 - P_n)TP_n\| \to 0$ $(n \to \infty)$.

Following this model, and [**15**], we shall say that $A \in \mathscr{L}(\mathscr{H})$ is *$(-m)$-triangular* if there exists $\{P_n\}_{n=0}^{\infty} \subset \mathscr{P}(\mathscr{H})$, with $P_n \uparrow 1$, such that $\operatorname{rank} P_n = n$ and $(1 - P_n)AP_{n+m} = 0$ for all $n = 0, 1, 2, \ldots$, that is, A admits a matrix representation of the form $A = (a_{ij})_{i,j=1}^{\infty}$ with $a_{ij} = 0$ for all $j < i + m$, with respect to some ONB of $\mathscr{H}$. A sequence $\{P_n\}_{n=0}^{\infty} \subset \mathscr{P}(\mathscr{H})$ satisfying the above conditions is obviously a *maximal chain* in $\mathscr{P}(\mathscr{H})$ and every maximal chain in $\mathscr{P}(\mathscr{H})$ converging strongly to 1 has that form.

An operator A will be called *almost $(-m)$-triangular* if there exists a maximal chain $\{P_n\}_{n=0}^{\infty} \subset \mathscr{P}(\mathscr{H})$, with$P_n \uparrow 1$, and a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $(1 - P_{n_k})AP_{n_k+m} = 0$ for all $k = 1, 2, \ldots$.

Clearly, every $(-m)$-triangular operator is almost $(-m)$-triangular, but the converse is false. (If R is an orthogonal projection of infinite rank and nullity, then R is almost $(-m)$-triangular for all $m \geqslant 1$, but R is $(-m)$-triangular for no value of m.)

Imitating the definition of quasitriangularity, we shall say that T is *strictly $(-m)$-quasitriangular* (class $(\mathrm{StrQT})_{-m}$) if $\|(1 - P_n)TP_{n+m}\| \to 0$ $(n \to \infty)$ for some maximal chain $\{P_n\}_{n=0}^{\infty} \subset \mathscr{P}(\mathscr{H})$, with $P_n \uparrow 1$, and a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $\|(1 - P_{n_k})TP_{n_k+m}\| \to 0$ $(k \to \infty)$ $(m = 1, 2, \ldots)$.

In the same vein, we define the *modulus of strict $(-m)$-quasitriangularity* $\mathrm{strq}_{-m}(R)$ of an operator $R \in \mathscr{L}(\mathscr{H})$ by

$$\mathrm{strq}_{-m}(R) = \inf\left\{ \limsup_n \|(1 - P_n)TP_{n+m}\| : \{P_n\}_{n=0}^{\infty} \subset \mathscr{P}(\mathscr{H}),\ P_n \uparrow 1, \right.$$
$$\left. \text{is a maximal chain} \right\},$$

and the *modulus of $(-m)$-quasitriangularity* $\mathrm{q}_{-m}(R)$, by

$$\mathrm{q}_{-m}(R) = \liminf_{P \in \mathscr{P}(\mathscr{H})} \left\{ \inf\|(1 - P)TP'\| : P' \subset \mathscr{P}(\mathscr{H}),\ P' > P, \operatorname{rank}(P' - P) = m \right\}.$$

We have the following chains of inclusions:

$$\begin{aligned}\cdots &\subset (\mathrm{StrQT})_{-(m+1)} \subset (\mathrm{StrQT})_{-m} \subset \cdots \subset (\mathrm{StrQT})_{-1} \subset (\mathrm{StrQT})_0\\ &= (\mathrm{QT}) \subset (\mathrm{StrQT})_1 \subset \cdots \subset (\mathrm{StrQT})_m \subset (\mathrm{StrQT})_{m+1} \subset \cdots,\\ \cdots &\subset (\mathrm{QT})_{-(m+1)} \subset (\mathrm{QT})_{-m} \subset \cdots \subset (\mathrm{QT})_{-1} \subset (\mathrm{QT})_0\\ &= (\mathrm{QT}) \subset (\mathrm{QT})_1 \subset \cdots \subset (\mathrm{QT})_m \subset (\mathrm{QT})_{m+1} \subset \cdots,\end{aligned}$$

and $(\mathrm{StrQT})_{-m} \subset (\mathrm{QT})_{-m}$ for all $m = 1, 2, \ldots,$ where $(\mathrm{StrQT})_m$ and $(\mathrm{QT})_m$ are the classes defined in **[15]**. (All the inclusions are proper, as we shall see immediately; $(\mathrm{QT})_{-m}$ is not included in $(\mathrm{StrQT})_{-p}$ for any $p > 0$.)

For the first restriction of quasitriangularity, we have the following result. (Recall that an operator A is *essentially normal* if $A^*A - AA^*$ is compact.)

THEOREM 1.2. *For each $m \geqslant 1$, the following statements are equivalent for T in $\mathscr{L}(\mathscr{H})$:*

$(\text{str-i})_{-m}$ $T \in (\mathrm{StrQT})_{-m}$.

$(\text{str-ii})_{-m}$ *There exist $\{P_n\}_{n=0}^{\infty}, \{R_n\}_{n=0}^{\infty} \subset \mathscr{P}(\mathscr{H})$, with $R_n \uparrow 1$, such that $\{R_n\}_{n=0}^{\infty}$ is a maximal chain, and $\|P_n - R_n\| \to 0$ and $\|(1 - P_n)TP_{n+m}\| \to 0$ $(n \to \infty)$.*

$(\text{str-iii})_{-m}$ $\operatorname{strq}_{-m}(T) = 0$.

$(\text{str-iv})_{-m}$ *T is the norm-limit of a sequence of operators of the form $A + F$, where A is $(-m)$-triangular and F has finite rank. (Furthermore, if $\sigma_0(T) = \varnothing$, then T is actually a norm-limit of (m)-triangular operators.)*

$(\text{str-v})_{-m}$ *The Weyl spectrum $\sigma_W(T) := \sigma_e(T) \cup \rho_{s\text{-}F}^{+}(T) \cup \rho_{s\text{-}F}^{-}(T)$ of T is a connected set containing the origin,* $\operatorname{ind}(\lambda - T) \geqslant 0$ *for all $\lambda \in \rho_{s\text{-}F}(T)$, and either T is not a semi-Fredholm operator, or T is semi-Fredholm with* $\operatorname{ind} T \geqslant m$.

$(\text{str-vi})_{-m}$ *T can be decomposed as $T = A + K$, where A is $(-m)$-triangular and K is compact. (Furthermore, if $\sigma_0(T) = \varnothing$, then given $\varepsilon > 0$, K can be chosen so that $\|K\| < \varepsilon$.)*

$(\text{str-ix})_{-m}$ *There exists a unilateral shift S_0, defined by $S_0 e_n = e_{n+1}$ with respect to an ONB $\{e_n\}_{n=1}^{\infty}$ of $\mathscr{H}$, such that $\|(1 - P_n)S_0^m TP_n\| \to 0$ $(n \to \infty)$, where P_n denotes the orthogonal projection of $\mathscr{H}$ onto the subspace $\bigvee\{e_1, e_2, \ldots, e_n\}$ spanned by the first n vectors of the basis $(n = 1, 2, \ldots)$.*

Furthermore, if $m \geqslant 2$, then each of the above statements is also equivalent to the following one:

$(\text{str-xi})_{-m}$ *For some (for all) j, $1 \leqslant j < m$, there exist $C_j \in \mathscr{K}(\mathscr{H})$ and a subspace $\mathscr{H}_j$ invariant under $T - C_j$, such that this operator admits a representation of the form*

$$T - C_j = \begin{pmatrix} T_j & L_j \\ 0 & A_j \end{pmatrix} \begin{matrix} \mathscr{H}_j \\ \mathscr{H} \ominus \mathscr{H}_j \end{matrix},$$

where $T_j \in \mathscr{L}(\mathscr{H}_j)$ is an essentially normal $(-j)$-triangular operator and $A_j \in \mathscr{L}(\mathscr{H} \ominus \mathscr{H}_j)$ is $(j - m)$-triangular; moreover, if $j \geqslant 2$, then T_j is unitarily equivalent to the direct sum of j essentially normal (-1)-triangular operators.

Furthermore,

$$\operatorname{dist}[R, (\mathrm{StrQT})_{-m}] = \operatorname{strq}_{-m}(R) \quad \textit{for all } R \in \mathscr{L}(\mathscr{H}).$$

The enumeration of the statements of Theorem 1.2 corresponds to the analogue enumeration of the statements of Theorem 1 of [**15**], to help with the comparison between two "symmetric" results. The absence of statements (str-vii)$_{-m}$, (str-viii)$_{-m}$, (str-viii)$_{-m,\varepsilon}$, (str-x)$_{-m}$, (str-xii)$_{-m}$, and (str-xii)$_{-m,\varepsilon}$ (analogous to the corresponding statements of [**15**, Theorem 1]) is due to the fact that we do not have, for $(-m)$-triangularity, a property symmetric to m-multiplicity. A similar observation applies to Theorem 1.2$^{\mathrm{a}}$ below. (Compare with Theorem 1$^{\mathrm{a}}$ of [**15**].)

For the second restriction of quasitriangularity, we have the following results.

THEOREM 1.2$^{\mathrm{a}}$. *For each $m \geqslant 1$, the following statements are equivalent for T in $\mathscr{L}(\mathscr{H})$:*

(i)$_{-m}$ $T \in (\mathrm{QT})_{-m}$.

(ii)$_{-m}$ *There exists $\{P_n\}_{n=0}^{\infty}$, $\{P_n'\}_{n=0}^{\infty} \subset \mathscr{P}(\mathscr{H})$ such that $P_n \leqslant P_n'$ and $\operatorname{rank}(P_n' - P_n) = m$ for all $n = 1, 2, \ldots$, $P_n \to 1$, $P_n' \to 1$ (strongly, but not necessarily increasingly) and $\|(1 - P_n)TP_n'\| \to 0$, as $n \to \infty$.*

(iii)$_{-m}$ $\mathrm{q}_{-m}(T) = 0$.

(iv)$_{-m}$ *T is the norm-limit of a sequence of almost $(-m)$-triangular operators.*

(v)$_{-m}$ *T is quasitriangular, and either $0 \in \sigma_{\mathrm{lre}}(T)$, or $0 \in \rho_{\mathrm{s\text{-}F}}^{+}(T)$ and $\operatorname{ind} T \geqslant m$.*

(vi)$_{-m}$ *$T = A + K$, where A is almost $(-m)$-triangular and K is compact.*

(vi)$_{-m,\varepsilon}$ *Given $\varepsilon > 0$, there exists $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, $\|K_\varepsilon\| < \varepsilon$, such that $A_\varepsilon = T - K_\varepsilon$ is almost $(-m)$-triangular.*

(ix)$_{-m}$ *$T \in (\mathrm{QT})$ and there exists a unilateral shift S of multiplicity one such that $S^m T \in (\mathrm{QT})$.*

(xi)$_{-m}$ *For some (for all) j, $1 \leqslant j \leqslant m$, there exist $C_j \in \mathscr{K}(\mathscr{H})$ and a subspace $\mathscr{H}_j$ invariant under $T - C_j$ such that this operator admits an operator matrix decomposition of the form*

$$T - C_j = \begin{pmatrix} T_j & L_j \\ 0 & A_j \end{pmatrix} \begin{matrix} \mathscr{H}_j \\ \mathscr{H} \ominus \mathscr{H}_j \end{matrix},$$

where $T_j \in \mathscr{L}(\mathscr{H}_j)$ is an essentially normal $(-j)$-triangular operator and $A_j \in \mathscr{L}(\mathscr{H} \ominus \mathscr{H}_j)$ is almost $(j - m)$-triangular. (In particular, if $j = m$ then A_j is triangular). Moreover, if $j \geqslant 2$, then T_j is unitarily equivalent to the direct sum of j essentially normal (-1)-triangular operators, and $\sigma_0(T_j) = \varnothing$.

(xi)$_{-m,\varepsilon}$ *Given $\varepsilon > 0$, for some (for all) j, $1 \leqslant j \leqslant m$, there exists $C_j \in \mathscr{K}(\mathscr{H})$, with $\|C_j\| < \varepsilon$, such that $T - C_j$ admits the representation of* (xi)$_{-m}$ *satisfying the conditions of that statement.*

Furthermore,

$$\operatorname{dist}[R, (\mathrm{QT})_{-m}] = \mathrm{q}_{-m}(R) \quad \text{for all } R \in \mathscr{L}(\mathscr{H}).$$

A large part of this article was developed during two informal seminars on operator theory held at the University of California at San Diego during the summers of 1983 and 1984 and (slightly weaker versions of) the results were announced in Chapter 13 of the monograph [**2**]. The author is deeply indebted to Professors J. Agler, J. A. Ball, L. C. Chadwick, L. A. Fialkow, J. W. Helton, and M. B. Pecuch for many helpful discussions.

2. On the formula 'quasitriangular − small compact = triangular'. Following C. Apostol [**1**], we define for T in $\mathscr{L}(\mathscr{H})$ and λ in $\rho_{\text{s-F}}(T)$,

$$\min\operatorname{ind}(\lambda - T) = \min\left\{\operatorname{nul}(\lambda - T), \operatorname{nul}(\lambda - T)^*\right\},$$

where $\operatorname{nul} R = \dim\ker R$.

We shall say that λ is a *regular point* of the semi-Fredholm domain of T (in symbols: $\lambda \in \rho^r_{\text{s-F}}(T)$) if the function $\mu \to \min\operatorname{ind}(\mu - T)$ is constant on some neighborhood of λ; $\rho^s_{\text{s-F}}(T) = \rho_{\text{s-F}}(t) \setminus \rho^r_{\text{s-F}}(T)$ is the set of all *singular points* of the semi-Fredholm domain of T.

Before establishing the most general result about the diagonal entries of a triangular compact perturbation of a quasitriangular operator, it is convenient to cite in full length two results of Apostol's article, which will be heavily used in the proofs.

Given $T \in \mathscr{L}(\mathscr{H})$, let $\mathscr{H}_r(T) = \bigvee\{\ker(\lambda - T)\colon \lambda \in \rho^r_{\text{s-F}}(T)\}$, let $\mathscr{H}_l(T) = \bigvee\{\ker(\lambda - T)^*\colon \lambda \in \rho^r_{\text{s-F}}(T)\}$, and let $\mathscr{H}_0(T)$ be the orthogonal complement of $\mathscr{H}_r(T) + \mathscr{H}_l(T)$. Denote the compression of T to $\mathscr{H}_r(T)$, $\mathscr{H}_l(T)$, and $\mathscr{H}_0(T)$ by T_r, T_l, and T_0, respectively.

THEOREM 2.1 [**1**, **11**, Theorem 3.38]. (i) *$\mathscr{H}_r(T)$ is orthogonal to $\mathscr{H}_l(T)$, so that $\mathscr{H} = \mathscr{H}_r(T) \oplus \mathscr{H}_0(T) \oplus \mathscr{H}_l(T)$.*

(ii) *$\mathscr{H}_r(T)$ and $\mathscr{H}_r(T) \oplus \mathscr{H}_0(T)$ are invariant under T, so that T admits a 3×3 upper triangular matrix representation*

$$T = \begin{pmatrix} T_r & * & * \\ 0 & T_0 & * \\ 0 & 0 & T_l \end{pmatrix}$$

with respect to the above decomposition, where

(iii) *$T_r = T \mid \mathscr{H}_r(T)$ is a triangular operator, $\sigma(T_r) = \sigma_l(T_r) = \sigma_{\text{lre}}(T_r) \cup \rho^+_{\text{s-F}}(T_r)$, $\sigma(T_r)$ is a perfect set, every component of $\sigma_{\text{lre}}(T_r)$ intersects the set $\sigma_p(T_r)^-$ and $\sigma_p(T_r^*) = \varnothing$, so that $\rho^s_{\text{s-F}}(T_r) = \varnothing$ and $\min\operatorname{ind}(\lambda - T_r) = 0$ for all $\lambda \in \rho_{\text{s-F}}(T_r)$, and*

(iv) *T_l is the adjoint in $\mathscr{L}(\mathscr{H}_l(T))$ of the triangular operator $T^* \mid \mathscr{H}_l(T)$, $\sigma(T_l) = \sigma_{\text{r}}(T_l) = \sigma_{\text{lre}}(T_l) \cup \rho^-_{\text{s-F}}(T_l)$, $\sigma(T_l)$ is a perfect set, every component of $\sigma_{\text{lre}}(T_l)$ intersects the set $[\sigma_p(T_l^*)^-]^*$, and $\sigma_p(T_l) = \varnothing$, so that $\rho^s_{\text{s-F}}(T_l) = \varnothing$ and*

$$\min\operatorname{ind}(\lambda - T_l) = 0 \quad \textit{for all } \lambda \in \rho_{\text{s-F}}(T_l).$$

Furthermore,

(v) *$\lambda \to P_{\ker(\lambda - T)}$ ($\lambda \in \rho_{\text{s-F}}(T)$) is a continuous function for λ in $\rho^r_{\text{s-F}}(T)$ and discontinuous for λ in $\rho^s_{\text{s-F}}(T)$;*

(vi) *$\rho_{\text{s-F}}(T) \subset \mathbf{C} \setminus [\sigma_r(T_r) \cup \sigma_l(T_l)]$;*

(vii) *$\rho^r_{\text{s-F}}(T) \subset \rho(T_0) := \mathbf{C} \setminus \sigma(T_0)$;*

(viii) *$\rho^s_{\text{s-F}}(T) \subset \sigma_0(T_0)$;*

(ix) *$\sigma_0(T) \subset \rho(T_r) \cap \rho(T_l) \cap \sigma_0(T_0)$;*

(x) *if Λ is a finite subset of $\rho^s_{\text{s-F}}(T)$, then T is similar to $T_\Lambda \oplus T'_\Lambda$, where T_Λ acts on a finite dimensional subspace, $\sigma(T_\Lambda) = \Lambda$ and $\Lambda \subset \rho^r_{\text{s-F}}(T'_\Lambda)$.*

In what follows, we shall write $A \sim B$ ($A \cong B$) to indicate that A and B are similar (unitarily equivalent, resp.) operators. If σ is a nonempty clopen subset of $\sigma(T)$, then $\mathscr{H}(\sigma; T)$ will denote the Riesz spectral subspace corresponding to σ (if $\sigma = \{\lambda\}$ is a singleton, we continue to write $\mathscr{H}(\lambda; T)$ to simplify the notation). For each subspace $\mathscr{M}$ of $\mathscr{H}$, $T_{\mathscr{M}}$ denotes the compression of T to $\mathscr{M}$; that is, $T_{\mathscr{M}} = P_{\mathscr{M}} T \mid \mathscr{M}$, where $P_{\mathscr{M}}$ denotes the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$. As usual, $A \oplus B$ will denote the direct sum of the operators $A \in \mathscr{L}(\mathscr{H}_1)$ and $B \in \mathscr{L}(\mathscr{H}_2)$, acting in the usual fashion on the orthogonal direct sum $\mathscr{H}_1 \oplus \mathscr{H}_2$ of the underlying spaces. For each cardinal α, $0 \leqslant \alpha \leqslant \aleph_0$, $A^{(\alpha)}$ is the direct sum of α copies of A, acting on the orthogonal direct sum $\mathscr{H}^{(\alpha)}$ of α copies of $\mathscr{H}$ ($A \in \mathscr{L}(\mathscr{H})$).

THEOREM 2.2 [**1**, **11**, Theorem 3.48; **12**]. *Let $T \in \mathscr{L}(\mathscr{H})$ and let $\varepsilon > 0$; then there exists $K \in \mathscr{K}(\mathscr{H})$ such that*

$$\|K\| < \varepsilon + \tfrac{1}{2} \max\{\operatorname{dist}[\lambda, \sigma_e(T)] : \lambda \in \sigma_0(T)\}$$

and

$$\min \operatorname{ind}(T + K - \lambda) = 0 \text{ for all } \lambda \in \rho_{s\text{-}F}(T).$$

In particular, $\sigma(T + K) = \sigma_W(T)$.

Indeed, ad hoc modifications of the proof of Theorem 2.2 (see [**1**, **11**, §3.4]) show that we can construct, for instance, $K \in \mathscr{K}(\mathscr{H})$, with $\|K\| < \varepsilon$, such that $T + K$ has the same normal eigenvalues as T, and K 'erases' part, but not all, of the singular points of $\rho_{s\text{-}F}^s(T) \setminus \sigma_0(T)$. It is not difficult to see that the points of $\sigma_0(T)$ cannot be removed with *small* compact perturbations, while all the points of $\rho_{s\text{-}F}^s(T) \setminus \sigma_0(T) = \rho_{s\text{-}F}^s(T) \cap$ [interior $\sigma(T)$] can be removed at once with an arbitrarily small compact perturbation. On the other hand, it is impossible to *add* a singularity at a regular point λ of $\rho_{s\text{-}F}(T)$ such that $\min \operatorname{ind}(\lambda - T) = 0$, by using a very small, not necessarily compact, perturbation. In this sense, $\{\lambda \in \rho_{s\text{-}F}(T)$: $\min \operatorname{ind}(\lambda - T) = 0\}$ is a highly stable part of $\rho_{s\text{-}F}(T)$ (under *small* compact perturbations).

For each h, $-\infty \leqslant h \leqslant \infty$, we write $\rho_{s\text{-}F}^h(T) = \{\lambda \in \rho_{s\text{-}F}(T)\colon \operatorname{ind}(\lambda - T) = h\}$.

Following C. Apostol [**1**, **11**, §3.4], we shall say that $T \in \mathscr{L}(\mathscr{H})$ is a *smooth* operator if

$$\min \operatorname{ind}(\lambda - T) = 0 \quad \text{for all } \lambda \in \rho_{s\text{-}F}(T).$$

If T is smooth, then $\sigma(T) = \sigma_W(T)$; in particular, $\rho_{s\text{-}F}^s(T) = \sigma_0(T) = \varnothing$.

Now we are in a position to state the main result of this article.

THEOREM 2.3. *Let T be a smooth quasitriangular operator, and let $\Gamma = \{\lambda_j\}_{j=1}^{\infty}$ be a sequence of complex numbers such that*

(i) $\lambda_j \in \sigma(T)$ *for all* $j = 1, 2, \ldots,$

and

(ii) *every nonempty clopen subset σ of $\sigma(T)$ satisfies* $\operatorname{card}\{j\colon \lambda_j \in \sigma\} = \aleph_0$.

Given $\varepsilon > 0$, there exists $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon\| < \varepsilon$, such that $A = T - K_\varepsilon$ is a smooth triangular operator with $d(A) = \Gamma$.

For arbitrary (i.e., not necessarily smooth) operators, we have the following general result:

COROLLARY 2.4. *Let* $T \in (\mathrm{QT})$, *and let* $\Gamma = \{\lambda_j\}_{j=1}^{\infty}$ *be a sequence of complex numbers such that*

(i) $\sigma_0(T) \subset \Gamma \subset \sigma_{\mathrm{lre}}(T) \cup \rho_{\text{s-F}}^{+}(T) \cup \sigma_0(T) \cup [\mathrm{interior}\{\rho_{\text{s-F}}^{0}(T) \cap \sigma(T)\}]$;

(ii) *every nonempty clopen subset* σ *of* $\sigma(T)$ *intersects* Γ, *and* $\mathrm{card}\{j\colon \lambda_j \in \sigma\} = \aleph_0$, *unless* σ *is a (necessarily finite) subset of* $\sigma_0(T)$;

(iii) $\mathrm{card}\{j\colon \lambda_j = \lambda\} = \dim \mathscr{H}(\lambda; T)$ *for each* $\lambda \in \sigma_0(T)$;

(iv) $\Gamma_0 = \{\lambda_j \in \Gamma\colon \lambda_j \in \mathrm{interior}[\rho_{\text{s-F}}^{0}(T) \cap \sigma(T)]\}$ *is a finite (possibly empty) or denumerable sequence whose limit points belong to* $\partial\sigma_e(T)$;

(v) Γ *includes a finite (possibly empty) or denumerable subsequence* Γ_s *such that every* λ *in* Γ_s *is a point of*

$$\left[\rho_{\text{s-F}}^{s}(T) \cap \left(\{\lambda \in \rho_{\text{s-F}}(T)\colon \min \mathrm{ind}(\lambda - T) = 0\}^{-}\right)\right] \setminus \sigma_0(T).$$

Moreover, if $\lambda \in \Gamma_s$, *and* $T \sim T_\lambda \oplus T_\lambda'$ *(with the notation of Theorem* 2.1(x)), *then* λ *is associated with some nonzero invariant subspace* $\mathscr{M}_\lambda$ *of* T_λ *(in a sense that will be made clear below), and* $\mathrm{card}\{\lambda_j \in \Gamma_s\colon \lambda_j = \lambda\} = \dim \mathscr{M}_\lambda$ *and*

(vi) Γ *also includes a finite (possibly empty) or denumerable subsequence*

$$\Gamma_s^0 \subset \rho_{\text{s-F}}(T) \cap \left(\{\lambda \in \rho_{\text{s-F}}^{r}(T)\colon \min \mathrm{ind}(\lambda - T) > 0\}^{-}\right),$$

Γ_s^0 *does not accumulate in* $\rho_{\text{s-F}}(T)$, *each* λ *in* Γ_s^0 *is associated with a given operator* R_λ *acting on a nonzero finite dimensional space* $\mathscr{R}_\lambda$, *and*

$$\mathrm{card}\left\{\lambda_j \in \rho_s^0\colon \lambda_j = \lambda\right\} = \dim \mathscr{R}_\lambda.$$

Then, for each $\varepsilon > 0$ *it is possible to find* $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, *with* $\|K_\varepsilon\| < \varepsilon$, *such that*

(1) $A = T - K_\varepsilon$ *admits a triangular matrix* $(a_{ij})_{i,j=1}^{\infty}$ *with respect to a suitable ONB* $\{e_j\}_{j=1}^{\infty}$ *of* $\mathscr{H}$;

(2) *the diagonal sequence* $d(A) = \{a_{jj}\}_{j=1}$ *of* A *coincides with the sequence* Γ;

(3) $\sigma(A) = \sigma_{\mathrm{lre}}(T) \cup \rho_{\text{s-F}}^{+}(T) \cup \sigma_0(T) \cup \{\lambda_j \in \Gamma_s^0\colon \mathrm{ind}(\lambda_j - T) = 0\}$;

(4) *if* $\Delta_\varepsilon = \{\lambda \in \sigma_0(T)\colon \mathrm{dist}[\lambda, \sigma_e(T)] > \varepsilon\}$, *then* $\mathscr{H}(\Delta_\varepsilon; A) = \mathscr{H}(\Delta_\varepsilon; T)$, *and* $A \mid \mathscr{H}(\Delta_\varepsilon; A) = T \mid \mathscr{H}(\Delta_\varepsilon; T)$;

(5) *if* $\lambda \in \sigma_0(T)$, *then* $\dim \mathscr{H}(\lambda; A) = \dim \mathscr{H}(\lambda; T)$, *and* $A \mid \mathscr{H}(\lambda; A) \sim T \mid \mathscr{H}(\lambda; T)$;

(6) *if* $\lambda \in \Gamma_s$, *then* $\lambda \in \rho_{\text{s-F}}^{s}(A)$, *and* $A \sim A_\lambda \oplus A_\lambda'$, *with* $A_\lambda \sim T_\lambda \mid \mathscr{M}_\lambda$; *and*

(7) *if* $\lambda \in \Gamma_s^0$, *then* $\lambda \in \rho_{\text{s-F}}^{s}(A)$ *and* $A \sim A_\lambda \oplus A_\lambda'$, *with* $A_\lambda \sim R_\lambda$.

Since the operator $A = T - K_\varepsilon$ of Corollary 2.4 is triangular, it readily follows from Proposition 1.1 that $\rho_{\text{s-F}}^{s}(A) = \sigma_0(T) \cup \Gamma_s \cup \Gamma_s^0$, and that the singular behavior associated with a singular point $\lambda \in \rho_{\text{s-F}}^{s}(A)$ is exactly the one described by (5)–(7) of that theorem. Comparison between Corollary 2.4 and Proposition 1.1 will convince the reader that Corollary 2.4 describes, essentially, the most general answer that we can expect for our problem. Of course, a small compact perturbation of T can produce, for instance, a triangular operator A such that $\sigma_0(A)$ is a finite subset of $\sigma_0(T)$ ('push' each of the points in $\sigma_0(T) \setminus \Delta_\varepsilon$ to a nearest point in $\partial\sigma_e(T)$!), or a triangular operator A all of whose normal eigenvalues have multiplicity one, etc.

However, the diagonal of any triangular operator obtainable from T by a small compact perturbation can now be easily described with the help of Theorem 2.2 and Corollary 2.4.

On the other hand, the most complete answer to the analogous problem for the case when we allow compact perturbations of arbitrarily large size is given by the following corollary. The proof is an (easy) exercise, and will be left to the reader. (Observe that, by Proposition 1.1, interior $\{\lambda \in \rho_{\text{s-F}}(A): \min \operatorname{ind}(\lambda - A) > 0\} = \varnothing$ for all triangular A.)

COROLLARY 2.5. *Let $T \in (\mathrm{QT})$, and let $\Gamma = \{\lambda_j\}_{j=1}^{\infty}$ be a sequence of complex numbers such that*

(a) *all the limit points of Γ belong to $\sigma_{\text{lre}}(T) \cup \rho_{\text{s-F}}^{+}(T)$, and*

(b) *if Ω is an open set such that $\Omega \cap [\sigma_{\text{lre}}(T) \cup \rho_{\text{s-F}}^{+}(T)] = \varnothing$, but $\partial\Omega \cap [\sigma_{\text{lre}}(T) \cup \rho_{\text{s-F}}^{+}(T)] = \varnothing$, then* $\operatorname{card}\{j: \lambda_j \in \Omega\} = \aleph_0$.

Then there exists $K \in \mathcal{K}(\mathcal{H})$ such that $A = T - K$ is triangular with diagonal sequence $d(A) = \Gamma$.

Conversely, if $C \in \mathcal{K}(\mathcal{H})$ and $B = T - C$ is triangular, then the diagonal sequence $d(B)$ of B necessarily satisfies (a) *and* (b).

In the remainder of this section, we shall reduce the proof of Corollary 2.4 to that of Theorem 2.3. First of all, it will be shown that the condition $d(A) = \Gamma$ (as *a sequence*) can be somehow relaxed. Indeed, the following lemma frees us from the tyranny of a particular ordering of the elements of Γ.

LEMMA 2.6. *Suppose that $A \in \mathcal{L}(\mathcal{H})$ admits a matrix representation of the form*

$$A = \begin{array}{c} \left(\begin{array}{cccccc} \lambda_1 & a_{12} & a_{13} & \cdots & a_{1j} & \cdots \\ & \lambda_2 & a_{23} & \cdots & a_{2j} & \cdots \\ & & \lambda_3 & \cdots & a_{3j} & \cdots \\ & & & \ddots & \vdots & \\ & & 0 & & \lambda_j & \\ & & & & & \ddots \end{array}\right) \end{array} \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_j \\ \vdots \end{array}$$

with respect to some ONB $\{e_j\}_{j=1}^{\infty}$ of $\mathcal{H}$, and let π be a permutation of the set $\mathbf{N}$ of all natural numbers (i.e., a bijective mapping of $\mathbf{N}$ onto itself). Then A also admits a representation of the form

$$A = \left(\begin{array}{cccccc} \lambda_{\pi(1)} & b_{12} & b_{13} & \cdots & b_{1j} & \vdots \\ & \lambda_{\pi(2)} & b_{23} & \cdots & b_{2j} & \vdots \\ & & \lambda_{\pi(3)} & \cdots & b_{3j} & \vdots \\ & & & \ddots & \vdots & \\ & & 0 & & \lambda_{\pi(j)} & \\ & & & & & \ddots \end{array}\right) \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_j \\ \vdots \end{array}$$

with respect to an ONB $\{f_j\}_{j=1}^{\infty}$ of $\mathcal{H}$.

PROOF. Define $g_1 = e_1$. It is easily seen that, for each $h > 1$, we can inductively find a nonzero vector g_h in

$$\left\{\bigvee\{e\}_{j=1}^{h} \cap \ker(\lambda_h - A)^h\right\} \setminus \bigvee\{g_1, g_2, \ldots, g_{h-1}\}.$$

Let $\{f_h\}_{h=1}^{\infty}$ be the Gram-Schmidt orthonormalization of the sequence $\{g_{\pi(h)}\}_{h=1}^{\infty}$, and let $\mathscr{R} = \bigvee\{f_h\}_{h=1}^{\infty}$. Clearly, $\mathscr{R}$ is invariant under A, and

$$A \mid \mathscr{R} = \begin{pmatrix} \lambda_{\pi(1)} & & & & & \\ & \lambda_{\pi(2)} & & & * & \\ & & \lambda_{\pi(3)} & & & \\ & & & \ddots & & \\ & 0 & & & \lambda_{\pi(h)} & \\ & & & & & \ddots \end{pmatrix} \begin{matrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_h \\ \vdots \end{matrix}.$$

It only remains to show that $\mathscr{R} = \mathscr{H}$; equivalently, we have to prove that $e_j \in \mathscr{R}$ for all $j = 1, 2, \ldots$.

Given j, let m be the first index such that $\{\pi(h)\}_{h=1}^{m} \supset \{1, 2, \ldots, j\}$; then

$$\mathscr{R} = \bigvee\{f_h\}_{h=1}^{\infty} \supset \bigvee\{f_h\}_{h=1}^{m} = \bigvee\{g_{\pi(h)}\}_{h=1}^{m} \supset \bigvee\{g_i\}_{i=1}^{j} = \bigvee\{e_i\}_{i=1}^{j} \ni e_j.$$

Hence, $\mathscr{R} = \mathscr{H}$. $\square$

First reduction. Let T, Γ and $\varepsilon > 0$ be as in Corollary 2.4. Given δ, $0 < \delta < \varepsilon/10$, let $\Delta_\delta = \{\lambda \in \sigma_0(T)\colon \operatorname{dist}[\lambda, \sigma_e(T)] > \delta\}$, and let $\mathscr{R}_\delta = \mathscr{H}(\Delta_\delta; T)$; then we have the decomposition

$$T = \begin{pmatrix} F_\delta & * \\ 0 & R_\delta \end{pmatrix} \begin{matrix} \mathscr{R}_\delta \\ \mathscr{H} \ominus \mathscr{R}_\delta \end{matrix} \sim F_\delta \oplus R_\delta,$$

where $F_\delta = T \mid \mathscr{R}_\delta$, and $R_\delta \sim T_\delta = T \mid \mathscr{H}(\sigma(T) \setminus \Delta_\delta; T)$. Clearly,

$$\rho_{\text{s-F}}^{s}(T_\delta) = \rho_{\text{s-F}}^{s}(T) \setminus \Delta_\delta,$$

and

$$\min \operatorname{ind}(\lambda - T_\delta)^k = \min \operatorname{ind}(\lambda - T)^k$$

for all $\lambda \in \rho_{\text{s-F}}(T) \setminus \Delta_\delta$, and for all $k = 1, 2, \ldots$, whence we deduce that the singular behavior of R_δ associated to each point μ of $\rho_{\text{s-F}}^{s}(R_\delta)$ is exactly the same as the singular behavior of T associated to μ (described by Theorem 2.1(vi)–(x)).

Let $\Sigma_\delta = \{\lambda \in \rho_{\text{s-F}}^{s}(T)\colon \lambda = \lambda_j$ for some λ_j in Γ_s, and $\operatorname{dist}[\lambda, \sigma_{\text{lre}}(T)] \geqslant \delta\}$.

By Theorem 2.1(x), there exists W invertible in $\mathscr{L}(\mathscr{H} \ominus \mathscr{R}_\delta)$ such that

$$R_\delta = W\left[R'_\delta \oplus \left\{\bigoplus_{\lambda \in \Sigma_\delta} T_\lambda\right\}\right] W^{-1},$$

where T_λ is defined as in Theorem 2.1(x), and $\rho_{\text{s-F}}^{s}(R'_\delta) = \rho_{\text{s-F}}^{s}(R_\delta) \setminus \Sigma_\delta$; then

$$T_\lambda = \begin{pmatrix} \lambda + Q_\lambda & * \\ 0 & \lambda + Q'_\lambda \end{pmatrix} \begin{matrix} \mathscr{M}_\lambda \\ \mathscr{M}_\lambda^{\perp} \end{matrix},$$

where Q_λ, Q'_λ are nilpotent operators acting on the finite dimensional spaces $\mathscr{M}_\lambda$ and, respectively, $\mathscr{M}_\lambda^{\perp}$.

For each $\lambda \in \Sigma_\delta$, we can find $\lambda'' \notin \rho^s_{\text{s-F}}(T)$ such that

$$|\lambda - \lambda''| < \delta/(\|W\| \cdot \|W^{-1}\|).$$

If

$$T''_\lambda = \begin{pmatrix} \lambda + Q_\lambda & * \\ 0 & \lambda'' + Q'_\lambda \end{pmatrix} \begin{matrix} \mathscr{M}_\lambda \\ \mathscr{M}_\lambda^\perp \end{matrix},$$

then $\|T_\lambda - T''_\lambda\| = |\lambda - \lambda''|$, $T'' \sim (\lambda + Q_\lambda) \oplus (\lambda'' + Q'_\lambda)$, and we can find a finite-rank operator K_1 such that $\mathscr{R}_\delta$ reduces K_1, $K_1 \mid \mathscr{R}_\delta = 0$,

$$\|K_1\| < \left(\max_{\lambda \in \Sigma_\delta} \|T_\lambda - T''_\lambda\|\right) \cdot \|W\| \cdot \|W^{-1}\| < \delta,$$

and

$$T - K_1 = \begin{pmatrix} F_\delta & * \\ 0 & W\left[R'_\delta \oplus \left\{\bigoplus_{\lambda \in \Sigma_\delta} T''_\lambda\right\}\right] W^{-1} \end{pmatrix} \begin{matrix} \mathscr{R}_\delta \\ \mathscr{H} \ominus \mathscr{R}_\delta \end{matrix}$$

$$= \begin{pmatrix} F_\delta & * & * \\ 0 & C(\Sigma_\delta) & * \\ 0 & 0 & S_\delta \end{pmatrix} \begin{matrix} \mathscr{R}_\delta \\ \mathscr{S}_\delta \\ \mathscr{H} \ominus (\mathscr{R}_\delta \oplus \mathscr{S}_\delta) \end{matrix},$$

where $C(\Sigma_\delta) \sim \bigoplus_{\lambda \in \Sigma_\delta} (\lambda + Q_\lambda)$, and $S_\delta \sim R'_\delta \oplus \{\bigoplus_{\lambda \in \Sigma_\delta} (\lambda'' + Q'_\lambda)\}$.

Observe that

$$\sigma_0(S_\delta) \subset \partial\sigma_{\text{lre}}(T)_\delta := \{\lambda \in \mathbf{C}: \operatorname{dist}[\lambda, \partial\sigma_{\text{lre}}(T)] \leqslant \delta\}.$$

By Theorem 2.2 and its proof (see [**1**, **11**, §3.4]), there exists K_2 compact, with $\|K_2\| < \delta$, such that $\mathscr{R}_\delta \oplus \mathscr{S}_\delta$ reduces K_2, $K_2 \mid \mathscr{R}_\delta \oplus \mathscr{S}_\delta = 0$, and

$$T - (K_1 + K_2) = \begin{pmatrix} F_\delta & * & * \\ 0 & C(\Sigma_\delta) & * \\ 0 & 0 & S'_\delta \end{pmatrix} \begin{matrix} \mathscr{R}_\delta \\ \mathscr{S}_\delta \\ \mathscr{H} \ominus (\mathscr{R}_\delta \oplus \mathscr{S}_\delta) \end{matrix},$$

where $\rho^s_{\text{s-F}}(S'_\delta) = \rho^s_{\text{s-F}}(T) \cap [\partial\sigma_{\text{lre}}(T)]_\delta$, and $\min\operatorname{ind}(\lambda - S'_\delta)^k = \min\operatorname{ind}(\lambda - T)^k$ for all $\lambda \in \rho_{\text{s-F}}(S'_\delta) \setminus \{\Delta_\delta \cup \Sigma_\delta\}$, and for all $k = 1, 2, \ldots$.

Second reduction. Let

$$S'_\delta = \begin{pmatrix} T_r & * & * \\ 0 & T_0 & * \\ 0 & 0 & T_l \end{pmatrix} \begin{matrix} \mathscr{H}_r(S'_\delta) \\ \mathscr{H}_0(S'_\delta) \\ \mathscr{H}_l(S'_\delta) \end{matrix}$$

be the triangular representation of S'_δ given by Theorem 2.1 (with T replaced by S'_δ), and let G_δ be a *cyclic* operator acting on a space of finite dimension m, such that $\sigma(G_\delta) \subset \Gamma^0_s \setminus [\partial\sigma_{\text{lre}}(T)]_\delta$.

By Theorem 2.2 and its proof (see the above references), we can find an invariant subspace $\mathscr{M}_\delta$ of T_r, $\dim \mathscr{M}_\delta = m$, such that

$$T_r = \begin{pmatrix} G'_\delta & * \\ 0 & T'_r \end{pmatrix} \begin{matrix} \mathscr{M}_\delta \\ \mathscr{H}_r(S'_\delta) \ominus \mathscr{M}'_\delta \end{matrix}, \quad \text{with } G'_\delta \sim G_\delta,$$

and K_3 compact, with $\|K_3\| < \delta$, such that

$$T-(K_1+K_2+K_3)=\left(\begin{array}{cccccc} F_\delta & * & * & * & * & * \\ 0 & C(\Sigma_\delta) & * & * & * & * \\ 0 & 0 & G'_\delta & * & * & * \\ 0 & 0 & 0 & T''_r & * & * \\ 0 & 0 & 0 & 0 & T''_0 & * \\ 0 & 0 & 0 & 0 & 0 & T''_l \end{array}\right)\begin{array}{l} \mathcal{R}_\delta \\ \mathcal{S}_\delta \\ \mathcal{M}_\delta \\ \mathcal{H}''_r(S'_\delta) \\ \mathcal{H}''_0(S'_\delta) \\ \mathcal{H}''_l(S'_\delta) \end{array},$$

where $\mathcal{H}''_r(S'_\delta) \subset [\mathcal{H}_r(S'_\delta) \ominus \mathcal{M}_\delta]$, $\mathcal{H}''_l(S'_\delta) \subset \mathcal{H}_l(S'_\delta)$, $\mathcal{H}''_0(S'_\delta) \supset \mathcal{H}_0(S'_\delta)$, and

$$\min\operatorname{ind}\left(\lambda - \begin{pmatrix} T''_r & * & * \\ 0 & T''_0 & * \\ 0 & 0 & T''_l \end{pmatrix}\right) = 0$$

for all $\lambda \in \rho_{s\text{-F}}(T)\setminus\rho^s_{s\text{-F}}(S'_\delta)$. (The subspace $\mathcal{H}''_r(S'_\delta) \oplus \mathcal{H}''_l(S'_\delta)$ reduces K_3, and $K_3 = 0$ on the orthogonal complement of this subspace.)

Now it is not difficult to see that, if G_δ is cleverly chosen, we can find a finite-rank operator $K_4 \in \mathcal{L}(\mathcal{H})$, with $\|K_4\| < \delta$, such that $\mathcal{M}_\delta$ reduces K_4, $K_4 \mid \mathcal{M}_\delta^\perp = 0$, and

$$\begin{aligned} &T-(K_1+K_2+K_3+K_4) \\ &= \left(\begin{array}{cccccc} F_\delta & F_{12,\delta} & F_{13,\delta} & * & * & * \\ 0 & C(\Sigma_\delta) & F_{23,\delta} & * & * & * \\ 0 & 0 & G''_\delta & * & * & * \\ 0 & 0 & 0 & T''_r & T_{12,\delta} & T_{13,\delta} \\ 0 & 0 & 0 & 0 & T''_0 & T_{23,\delta} \\ 0 & 0 & 0 & 0 & 0 & T''_l \end{array}\right)\begin{array}{l} \mathcal{R}_\delta \\ \mathcal{S}_\delta \\ \mathcal{M}_\delta \\ \mathcal{H}''_r(S'_\delta) \\ \mathcal{H}''_0(S'_\delta) \\ \mathcal{H}''_0(S'_\delta) \end{array}, \end{aligned}$$

where $G''_\delta \sim \oplus\{R_\lambda\colon \lambda \in \Gamma^0_s\setminus[\partial\sigma_{\text{lre}}(T)]_\delta\}$. (The details of the choice of the right G_δ and the right K_4 are straightforward and will be left to the interested reader, if any!)

Thus

$$T-(K_1+K_2+K_3+K_4) = \begin{pmatrix} C_\delta & C'_\delta \\ 0 & A_\delta \end{pmatrix}\begin{array}{l} \mathcal{H}\ominus\mathcal{H}_\delta \\ \mathcal{H}_\delta \end{array},$$

where $\mathcal{H}\ominus\mathcal{H}_\delta = \mathcal{R}_\delta \oplus \mathcal{S}_\delta \oplus \mathcal{M}_\delta$ is a finite dimensional subspace,

$$C_\delta = \begin{pmatrix} F_\delta & F_{12,\delta} & F_{13,\delta} \\ 0 & C(\Sigma_\delta) & F_{23,\delta} \\ 0 & 0 & G''_\delta \end{pmatrix}\begin{array}{l} \mathcal{R}_\delta \\ \mathcal{S}_\delta \\ \mathcal{M}_\delta \end{array}, \quad \text{and} \quad A_\delta = \begin{pmatrix} T''_r & T_{12,\delta} & T_{13,\delta} \\ 0 & T''_0 & T_{23,\delta} \\ 0 & 0 & T''_l \end{pmatrix}\begin{array}{l} \mathcal{H}''_r(S'_\delta) \\ \mathcal{H}''_0(S'_\delta) \\ \mathcal{H}''_l(S'_\delta) \end{array};$$

moreover, $T-(K_1+K_2+K_3+K_4) \sim C_\delta \oplus A_\delta$, $\rho^s_{s\text{-F}}(A_\delta) = \rho^s_{s\text{-F}}(T) \cap [\partial\sigma_{\text{lre}}(T)]_\delta$,

$$\min\operatorname{ind}(\lambda - A_\delta) = 0 \quad \text{for all } \lambda \in \rho^r_{s\text{-F}}(A_\delta),$$

and

$$\min \operatorname{ind}(\lambda - A_\delta)^k - \min \operatorname{ind}(\lambda - A_\delta)^{k-1}$$
$$= \min \operatorname{ind}(\lambda - T)^k - \min \operatorname{ind}(\lambda - T)^{k-1}$$

for all $\lambda \in \rho^s_{\text{s-F}}(A_\delta)$, and for all $k = 1, 2, 3, \ldots$.

Third reduction. Since $\rho^s_{\text{s-F}}(A_\delta) \subset [\partial\sigma_{\text{lre}}(T)]_\delta = [\partial\sigma_{\text{lre}}(A)]_\delta$, we can use J. G. Stampfli's argument in order to find a compact normal operator $K_5' \in \mathscr{L}(\mathscr{H}_\delta)$, with $\|K_5'\| \leqslant \delta$, such that $A_\delta - K_5'$ is a smooth operator. (Roughly speaking, K_5' 'pushes' each λ in $\rho^s_{\text{s-F}}(A_\delta)$ to one of its nearest points in $\partial\sigma_{\text{lre}}(A_\delta)$; see **[22]**, or **[11**, Proposition 3.45] for details.)

Fourth reduction. Let $\Gamma = \{\lambda_j\}_{j=1}^\infty$ be the sequence of Corollary 2.4, and let $\Gamma' = \{\lambda_j'\}_{j=1}^\infty$ be the subsequence obtained from Γ after deleting the finitely many terms corresponding to the diagonal entries of a triangular matrix representation of the operator C_δ. To each λ_j' in Γ' we associate a point $\mu_j' \in \sigma(A_\delta) \setminus \sigma_0(A_\delta)$, according to the following rules. Let Γ_δ be the subsequence of Γ consisting of all those λ_j's such that $\lambda_j \in \sigma_0(A_\delta)(\Gamma_\delta \subset \Gamma'!)$; then:

(1) if $\lambda_j' \in \Gamma_\delta \cup \Gamma_s \cup \Gamma_s^0$, we define μ_j' as some point in $\partial\sigma_{\text{lre}}(A_\delta)$ $(= \partial\sigma_{\text{lre}}(T))$ such that $|\lambda_j' - \mu_j'| = \operatorname{dist}[\lambda_j', \sigma_{\text{lre}}(T)]$;

(2) if $\lambda_j' \notin \Gamma_\delta \cup \Gamma_s \cup \Gamma_s^0$, we set $\mu_j' = \lambda_j'$.

Let $\Xi = \{\mu_j'\}_{j=1}^\infty$. Assume that Theorem 2.3 is true; then we can combine this result with Stampfli's construction **[22]** in order to find an operator $K_6' \in \mathscr{K}(\mathscr{H}_\delta)$, with $\|K_6'\| < 2\delta$, such that $A_\delta - K_6'$ is a smooth triangular operator with matrix $(a_{ij}')_{i,j=1}^\infty$ such that $d(A_\delta - K_6') = \{a_{jj}'\}_{j=1}^\infty$ coincides with Ξ. (Take $K_6' = K_6'(1) + K_6'(2)$, where $K_6'(1)$ is a compact normal operator that 'pushes' λ_j' to μ_j' for all $\lambda_j' \in \Gamma_\delta \cup \Gamma_s \cup \Gamma_s^0$, and then use Theorem 2.3 to construct a suitable $K_6'(2)$, with $\|K_6'(1)\| < \delta$ and $\|K_6'(2)\| < \delta$.)

Fifth reduction. Now, with the help of Theorem 2.2, we can construct $K_7' \in \mathscr{K}(\mathscr{H}_\delta)$, with $\|K_7'\| < 2\delta$, which 'pulls' μ_j' back to λ_j' for all $\lambda_j' \in \Gamma_\delta \cup \Gamma_s \cup \Gamma_s^0$, in such a way that

$A_\delta'' = A_\delta - (K_6' + K_7') = (a_{ij}'')_{i,j=1}^\infty$,

$a_{ij}'' = a_{ij}'$, if $i \neq j$ $(= 0$ if $i > j)$, or $i = j$ but $\lambda_j' \notin \Gamma_\delta \cup \Gamma_s \cup \Gamma_s^0$,

$a_{jj}'' = \lambda_j'$, if $\lambda_j' \in \Gamma_\delta \cup \Gamma_s \cup \Gamma_s^0$, $\rho^s_{\text{s-F}}(A_\delta'') = \{\lambda_j' \colon \lambda_j' \in \Gamma_\delta \cup \Gamma_s \cup \Gamma_s^0\}$, and

$A_\delta'' \sim A_\delta''(\lambda) \oplus \lambda(\mathbf{C}^{m(\lambda)})$ for each $\lambda \in \mathbf{C}$ such that $\lambda = \lambda_j'$ for some λ_j' in $\Gamma_\delta \cup \Gamma_s \cup \Gamma_s^0$, where $\lambda \notin \rho^s_{\text{s-F}}(A_\delta''(\lambda))$, $\lambda(\mathbf{C}^{m(\lambda)})$ is 'λ times the identity operator' on $\mathbf{C}^{m(\lambda)}$, and

$$m(\lambda) = \begin{cases} \dim \mathscr{H}(\lambda; T), & \text{if } \lambda \in \Gamma_\delta, \\ \dim \mathscr{M}_\lambda, & \text{if } \lambda \in \Gamma_s, \\ \dim \mathscr{R}_\lambda, & \text{if } \lambda \in \Gamma_s^0. \end{cases}$$

Sixth reduction. Finally, by a straightforward construction (by modifying 'point by point' in $\rho^s_{\text{s-F}}(A_\delta'')$) we can find $K_8' \in \mathscr{K}(\mathscr{H}_\delta)$, with $\|K_8'\| < \delta$, such that the operator $B_\delta = A_\delta'' - K_8'$ admits a triangular matrix $(b_{ij})_{i,j=1}^\infty$ (with respect to a suitable ONB of $\mathscr{H}_\delta$) with diagonal sequence $d(B_\delta) = \{b_{jj}\}_{j=1}^\infty = d(A_\delta'') = \Gamma'$,

$\rho^s_{\text{s-F}}(B_\delta) = \rho^s_{\text{s-F}}(A''_\delta) = \{\lambda'_j : \lambda'_j \in \Gamma_\delta \cup \Gamma_s \cup \Gamma^0_s\}$, and $B_\delta \sim B'_\delta(\lambda) \oplus B_\delta(\lambda)$ for each λ in $\rho^s_{\text{s-F}}(B_\delta)$, where $\min \operatorname{ind}(\lambda - B'_\delta(\lambda)) = 0$, and

$$B_\delta(\lambda) \text{ is similar to } \begin{cases} T \mid \mathscr{H}(\lambda; T), & \text{if } \lambda \in \Gamma_\delta, \\ T_\lambda \mid \mathscr{M}_\lambda, & \text{if } \lambda \in \Gamma_s, \\ R_\lambda, & \text{if } \lambda \in \Gamma^0_s. \end{cases}$$

Let $K_\varepsilon = \sum_{r=1}^8 K_r$, where $K_r = O_{\mathscr{H} \ominus \mathscr{H}_\delta} \oplus K'_r$ for $r = 5, 6, 7, 8$; then $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, $\|K_\varepsilon\| < 10\delta < \varepsilon$, and

$$A = T - K_\varepsilon = \begin{pmatrix} C_\delta & C'_\delta \\ 0 & B_\delta \end{pmatrix} \begin{matrix} \mathscr{H} \ominus \mathscr{H}_\delta \\ \mathscr{H}_\delta \end{matrix} \sim C_\delta \oplus B_\delta.$$

Since $d(B_\delta) = \Gamma'$ and C_δ has a triangular matrix whose diagonal entries coincide with the finite sequence $(\Gamma \setminus \Gamma')$, it follows from Lemma 2.6 that A admits a triangular matrix $(a_{ij})_{i,j=1}$ with diagonal sequence $d(A) = \{a_{jj}\}_{j=1}^\infty = \Gamma$; i.e., A has the properties (1) and (2) of Corollary 2.4. Furthermore, it is not difficult to check from the above construction that A also has properties (3)–(7) (use Proposition 1.1).

Thus, we have reduced (in a very concrete way!) the proof of Corollary 2.4 to that of Theorem 2.3.

3. 'Smooth quasitriangular − small compact = smooth triangular'. This section is devoted to proving Theorem 2.3.

LEMMA 3.1. *Let $T \in \mathscr{L}(\mathscr{H})$, and let $\Omega = \operatorname{interior} \Omega^-$ be a nonempty bounded open set such that $\partial\Omega \subset \sigma_{\text{le}}(T)$ and $\Omega^- \subset \sigma_{\text{lre}}(T) \cup \sigma_p(T)$. Given $\varepsilon > 0$ there exist $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon\| < \varepsilon$, and an essentially normal A such that $\sigma(A) = \Omega^-$, $\sigma_e(A) = \partial\Omega$, $\operatorname{ind}(\lambda - A) = \operatorname{nul}(\lambda - A) = 1$ for all $\lambda \in \Omega$, and*

$$T - K_\varepsilon = \begin{pmatrix} A & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

satisfies

$$\min \operatorname{ind}(T - K_\varepsilon - \lambda)^k = \min \operatorname{ind}(T - \lambda)^k$$

for all $\lambda \in \rho_{\text{s-F}}(T)$, and for all $k = 1, 2, \ldots$.

PROOF. It is not difficult to check that Ω^- is actually a subset of $\sigma_{\text{lre}}(T) \cup \sigma_p(T_r)$, where $T_r = T \mid \mathscr{H}_r(T)$ (as in Theorem 2.1).

By Voiculescu's theorem, there exists $K_1 \in \mathscr{K}(\mathscr{H})$, with $\|K_1\| < \varepsilon/4$, such that

$$T - K_1 \simeq T \oplus \begin{pmatrix} N & B_{12} \\ 0 & B_{22} \end{pmatrix},$$

where N is a given normal operator such that $\sigma(N) = \sigma_e(N) = \partial\Omega$, and $\sigma(B_{22}) \subset \sigma_e(T)$ (see [**11**, Chapter 4; **23**]; the precise form of N will be described below).

Let $\{\Omega_i\}$ be an enumeration of the components of Ω; for each i, choose $\omega_i \in \Omega_i$, and μ_i a representing measure for ω_i supported by $\partial\Omega_i$ [7]. Now we define $N = M(\partial\Omega)^{(\infty)}$, where $M(\partial\Omega) =$ 'multiplication by λ' acting on $L^2(\partial\Omega, \mu)$, where

$\mu = \Sigma_i 2^{-i}\mu_i$. Thus,

$$M(\partial\Omega) = \begin{pmatrix} M_+(\partial\Omega) & Z \\ 0 & M_-(\partial\Omega) \end{pmatrix} \begin{matrix} H^2(\partial\Omega) \\ L^2(\partial\Omega,\mu) \ominus H^2(\partial\Omega) \end{matrix},$$

(where $H^2(\partial\Omega)$ denotes the closure in $L^2(\partial\Omega, \mu)$ of the rational functions with poles outside Ω^-), and (by Theorem 2.1)

$$T - K_1 \simeq \begin{pmatrix} T_r & * & * \\ 0 & T_0 & * \\ 0 & 0 & T_l \end{pmatrix} \oplus \begin{pmatrix} M_+(\partial\Omega) & \begin{pmatrix} Z & 0 \end{pmatrix} & * & * \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} M_-(\partial\Omega) & 0 \\ 0 & N^{(2)} \end{pmatrix} & \begin{matrix} * \\ * \end{matrix} & \begin{matrix} * \\ * \end{matrix} \\ 0 & \begin{matrix} 0 & 0 \end{matrix} & N & * \\ 0 & \begin{matrix} 0 & 0 \end{matrix} & 0 & B'_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} M_-(\partial\Omega) & 0 \\ 0 & N \end{pmatrix} & 0 & * \\ \begin{pmatrix} Z & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} M_+(\partial\Omega) & 0 & 0 \\ 0 & T_r & 0 \\ 0 & 0 & N \end{pmatrix} & * \\ 0 & 0 & \begin{pmatrix} T_0 & * \\ 0 & T_l \end{pmatrix} \oplus \begin{pmatrix} N & * \\ 0 & B'_{22} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} V & 0 & * \\ Z' & R & * \\ 0 & 0 & T_l'' \end{pmatrix},$$

where

$$V = M_-(\partial\Omega) \oplus N, \quad Z' = \begin{pmatrix} Z & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad R = M_+(\partial\Omega) \oplus T_r \oplus N,$$

and

$$T_l'' = \begin{pmatrix} T_0 & * \\ 0 & T_l \end{pmatrix} \oplus \begin{pmatrix} N & * \\ 0 & B'_{22} \end{pmatrix} \qquad \bigl(\sigma(B'_{22}) = \sigma(B_{22})\bigr).$$

Observe that R is quasitriangular, $\sigma(R) \subset \sigma(T) \setminus \sigma_0(T)$, and $\sigma_0(R) = \varnothing$; therefore, we can find $K_2 \in \mathscr{K}(\mathscr{H})$, $K_2 \simeq 0 \oplus K_2' \oplus 0$, with $\|K_2\| < \varepsilon/4$, such that

$$T - (K_1 + K_2) \simeq \begin{pmatrix} V & 0 & * \\ Z' & R' & * \\ 0 & 0 & T_l'' \end{pmatrix} \begin{matrix} \mathscr{H}_1 \\ \mathscr{H}_2, \\ \mathscr{H}_3 \end{matrix}$$

where $R' = R - K_2'$ is triangular with respect to an ONB $\{f_j\}_{j=1}^\infty$ of $\mathscr{H}_2$, with $d(R') = \{r_{jj}\}_{j=1}^\infty$, and $\sigma(R') \subset \sigma_{\mathrm{lre}}(T)$ (use [**15**, Proposition 4.4, **2**, Proposition 13.11]).

Let P_n denote the orthogonal projection of $\mathscr{H}_2$ onto $\bigvee\{f_1, f_2, \ldots, f_n\}$. Since Z' is compact [**3**, **11**, §4.1], we can find $n = n(\varepsilon)$ so that $\|(1 - P_n)Z'\| < \varepsilon/4$. For this n, we have

$$T - (K_1 + K_2 + K_3) \simeq \begin{array}{c} \begin{pmatrix} V & 0 & 0 & * \\ P_n Z' & R'_n & * & * \\ 0 & 0 & R'' & * \\ 0 & 0 & 0 & T''_l \end{pmatrix} \end{array} \begin{array}{l} \mathscr{H}_0 \\ \mathscr{H}'_2, \\ \mathscr{H}_3 \end{array}$$

where $\mathscr{H}_0 = \mathscr{H}_1 \oplus \operatorname{ran} P_n$, $\mathscr{H}'_2 = \mathscr{H}_2 \ominus \operatorname{ran} P_n$, $R'' = R'_{\mathscr{H}'_2}$, $R'_n = R' \,|\operatorname{ran} P_n$, and

$$K_3 \simeq \begin{pmatrix} 0 & 0 \\ (1 - P_n)Z' & 0 \\ 0 & 0 \end{pmatrix}$$

is compact, with $\|K_3\| < \varepsilon/4$.

Clearly,

$$W = \begin{pmatrix} V & * \\ P_n Z' & R'_n \end{pmatrix}$$

is a quasitriangular finite-rank perturbation of $V \oplus 1_{\operatorname{ran} P_n}$. In particular, $M_+(\partial\Omega)$, V and W are essentially normal operators [**3**, **11**, §4.1] and, if n is large enough, $\sigma_0(W) \subset \partial\Omega_{\varepsilon/4}$ (use the upper semicontinuity of the spectrum [**11**, Chapter 1, **19**]). By using Stampfli's argument [**22**], we can find a normal compact operator $K_4 \simeq K'_4 \oplus 0_{\mathscr{H}'_2} \oplus 0_{\mathscr{H}_3}$, with $\|K_4\| \leqslant \varepsilon/4$; such that $A = W - K_4$ satisfies our requirements, and

$$T - K_\varepsilon \simeq \begin{pmatrix} A & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

(where $K_\varepsilon = K_1 + K_2 + K_3 + K_4 \in \mathscr{K}(\mathscr{H})$, and $\|K_\varepsilon\| < \varepsilon$) satisfies the conditions

$$\min\operatorname{ind}(T - K_\varepsilon - \lambda)^k = \min\operatorname{ind}(T - \lambda)^k$$

for all $\lambda \in \rho_{\text{s-F}}(T)$, and for all $k = 1, 2, \ldots$.

Indeed, by construction, both A and R'' are smooth quasitriangular operators, whence it readily follows from Theorem 2.1 that

$$L(B) = \begin{pmatrix} A & B \\ 0 & R'' \end{pmatrix}$$

is also a smooth quasitriangular operator for all B, because $\ker[\lambda - L(B)]^*$ is necessarily trivial for all λ in $\rho_{\text{s-F}}(T)$. Thus, the only singularities of $T - K_\varepsilon$ are those 'accumulated' in the T_0 entry of T''_l, whence the result follows. □

The second ingredient of the proof is a finite dimensional approximation argument.

LEMMA 3.2. *Let γ be a Jordan arc in $\mathbf{C}$, joining ν and μ, and let $R \in \mathscr{L}(\mathbf{C}^m)$ be an operator such that $\sigma(R) \subset \gamma$. Given $\varepsilon > 0$, there exist $n = n(\gamma, \varepsilon, R)$, a normal operator $N \in \mathscr{L}(\mathbf{C}^n)$ such that $\mu, \nu \in \sigma(N) \subset \gamma$, and a nilpotent operator $Q \in \mathscr{L}(\mathbf{C}^{m+n})$ such that $\|R \oplus N - (\nu + Q)\| < \varepsilon$.*

PROOF. If $m = 1$, this is [**12**, Lemma 2.7] (or [**11**, Lemma 12.45]).

Assume $m \geqslant 2$. Clearly, we can find m distinct points $\mu_1, \mu_2, \ldots, \mu_m \in \gamma$ such that $\operatorname{dist}[\mu_j, \sigma(R)] < \varepsilon/2$ for all $j = 1, 2, \ldots, m$, and an operator $R' \in \mathscr{L}(\mathbf{C}^m)$ such that $\sigma(R') = \{\mu_1, \mu_2, \dots, \mu_m\}$ and

$$\|R - R'\| = \max\{\operatorname{dist}[\mu_j, \sigma(R)] : 1 \leqslant j \leqslant m\} < \varepsilon/2.$$

Since R' has m distinct eigenvalues, there exists W invertible in $\mathscr{L}(\mathbf{C}^m)$ such that $R' = W(\operatorname{diag}\{\mu_1, \mu_2, \dots, \mu_m\})W^{-1}$. By applying the result of the above references to the $m + 1$ points $\mu_0 = \mu, \mu_1, \mu_2, \dots, \mu_m$, with ε replaced by $(\varepsilon/2)(\|W\| \cdot \|W^{-1}\|)$, we can find $p_j = p_j(\gamma, \varepsilon, \mu_j)$ and $N_j', Q_j' \in \mathscr{L}(\mathbf{C}^{p_j+1})$ such that $N_j' = \operatorname{diag}\{\nu, \mu_{j,2}, \mu_{j,3}, \dots, \mu_{j,p_j}, \mu_j\}$, $\sigma(N_j') \subset \gamma$, Q_j' is nilpotent, and

$$\left\|N_j' - \left(\nu + Q_j'\right)\right\| < (\varepsilon/2)(\|W\| \cdot \|W^{-1}\|), \qquad j = 0, 1, 2, \dots, m.$$

Let $n = 1 + \sum_{j=0}^m p_j$, $N_j = \operatorname{diag}\{\nu, \mu_{j,2}, \mu_{j,3}, \dots, \mu_{j,p_j}\}$, $N = N_0' \oplus \{\oplus_{j=1}^m N_j\}$, and $Q' = \oplus_{j=0}^m Q_j'$.

Let

$$Q = \left(1_{p_0+1} \oplus 1_{p_1} \oplus 1_{p_2} \oplus \cdots \oplus 1_{p_m} \oplus W\right) Q' \left(1_{p_0+1} \oplus 1_{p_1} \oplus 1_{p_2} \oplus \cdots \oplus 1_{p_m} \oplus W\right)^{-1},$$

where 1_{p_0+1} = identity on the space of N_0', 1_{p_j} = identity on the space of N_j ($j = 1, 2, \dots, m$), and W acts on the space $\mathbf{C}^m$ of R'', suitably identified with the space of the 'last coordinates' of N_j' ($j = 1, 2, \dots, m$); then $Q \in \mathscr{L}(\mathbf{C}^{m+n})$ is nilpotent, $N \in \mathscr{L}(\mathbf{C}^n)$ is normal, and

$$\begin{aligned}
\|N \oplus R - (\nu + Q)\| &\leqslant \|N \oplus R - N \oplus W(\operatorname{diagonal}\{\mu_1, \mu_2, \dots, \mu_m\})W^{-1}\| \\
&\quad + \|N \oplus W(\operatorname{diag}\{\mu_1, \mu_2, \dots, \mu_m\})W^{-1} - (\nu + Q)\| \\
&\leqslant \|R - R'\| + \|W\| \cdot \left\| \bigoplus_{j=0}^{m} \left[N_j' - \left(\nu + Q_j'\right)\right] \right\| \cdot \|W^{-1}\| \\
&< \varepsilon/2 + (\varepsilon/2)(\|W\| \cdot \|W^{-1}\|)\|W\| \cdot \|W^{-1}\| = \varepsilon. \qquad \square
\end{aligned}$$

LEMMA 3.3. *Let $T \in \mathscr{L}(\mathscr{H})$ be a smooth quasitriangular operator, let $N \in \mathscr{L}(\mathscr{R})$ be a diagonal normal operator such that $\sigma(N) = \sigma_e(N) = \sigma_e(T)$, and let $\mathscr{M}$ be a finite dimensional subspace of $\mathscr{H}$.*

Suppose that $0 \in \sigma(T)$, and $\sigma(T)_\varepsilon$ is a connected set (for some ε, $0 < \varepsilon < 1$); then there exist $K_\varepsilon \in \mathscr{K}(\mathscr{H} \oplus \mathscr{R} \oplus \mathscr{R})$, with $\|K_\varepsilon\| < (100 + 10\|T\|)\varepsilon$, and a finite dimensional subspace $\mathscr{N}$ of $\mathscr{H} \oplus \mathscr{R} \oplus \mathscr{R}$ including $\mathscr{M}$, invariant under $T \oplus N \oplus N - K_\varepsilon$, such that the spectrum of $(T \oplus N \oplus N - K_\varepsilon)|\mathscr{N}$ is the singleton $\{0\}$, and both $T \oplus N \oplus N - K_\varepsilon$ and its compression to $\mathscr{H} \ominus \mathscr{N}$ are smooth quasitriangular operators.

PROOF. *Preparation.* By using the upper semicontinuity of the spectrum, we can find $C_1 \in \mathscr{K}(\mathscr{H})$, with $\|C_1\| < \varepsilon$, such that $T - C_1$ has a triangular matrix (with respect to a suitable ONB) and $\sigma(T - C_1) = \sigma(T)$.

Let $\mathscr{N}_0$ be the linear span of the first n vectors of the basis. Clearly, if n is large enough, then

$$\|P_{\mathscr{M}} - P_{\mathscr{N}_0} P_{\mathscr{M}} P_{\mathscr{N}_0}\| < \varepsilon.$$

Assume that this is, indeed, the case, and consider the decomposition

$$T - C_1 = \begin{pmatrix} T_{11} & * \\ 0 & T_{22} \end{pmatrix} \begin{matrix} \mathscr{N}_0 \\ \mathscr{H} \ominus \mathscr{N}_0 \end{matrix}.$$

The operator T_{22} is also quasitriangular, and satisfies $\sigma(T_{22}) = \sigma_W(T_{22}) = \sigma(T)$. By Theorem 2.2, we can find $C_2 \in \mathscr{L}(\mathscr{H} \ominus \mathscr{N}_0)$, with $\|C_2\| < \varepsilon$, such that $T_{22} - C_2$ is smooth; furthermore, by Theorem 2.2 and Lemma 3.1, there exists $C_3 \in \mathscr{L}(\mathscr{H} \ominus \mathscr{N}_0)$, with $\|C_3\| < \varepsilon$, such that, if Ω denotes the interior of $\sigma(T)$, then

$$T_{22} - (C_2 + C_3) = \begin{array}{c}\begin{pmatrix} A & * \\ 0 & T'_{22} \end{pmatrix}\end{array}\begin{array}{l}\mathscr{H}_1\\ \mathscr{H}_2\end{array},$$

where A is an essentially normal operator such that $\sigma(A) = \Omega^-$, $\sigma_e(A) = \partial\Omega$ and $\operatorname{ind}(\lambda - A) = \operatorname{nul}(\lambda - A) = 1$ for all $\lambda \in \Omega$, and T'_{22} is smooth.

Let $\mathscr{R}_1$ ($\mathscr{R}_2$) denote the first (resp., the second) copy of the space $\mathscr{R}$; we have

$$\begin{aligned} T' &= \left[T - C_1 - 0_{\mathscr{N}_0} \oplus (C_2 + C_3)\right] \oplus N \oplus N \\ &= \begin{pmatrix} T_{11} & * & * \\ 0 & (A \oplus N) \oplus N & * \\ 0 & 0 & T'_{22} \end{pmatrix}\begin{array}{l}\mathscr{N}_0\\ (\mathscr{H}_1 \oplus \mathscr{R}_1) \oplus \mathscr{R}_2.\\ \mathscr{H}_2\end{array}\end{aligned}$$

Let Σ be the closure of an open neighborhood of $\sigma_e(T)_\varepsilon$ included in $\sigma_e(T)_{2\varepsilon}$, such that $\partial\Sigma$ consists of finitely many pairwise disjoint analytic Jordan curves. Clearly, Σ has only finitely many components, and $\Omega \setminus \Sigma$ has finitely many components. We can assume, moreover, that Σ has been chosen so that $\Omega \cup \Sigma$ is connected. We can find an analytic Jordan arc γ: $[0,1] \to \mathbf{C}$ such that $\gamma(0) = 0$, $\gamma(1) \in \partial\Sigma \cap \partial(\Omega \setminus \Sigma)$, and γ intersects all the components of Σ, and all the components of $\Omega \setminus \Sigma$.

We shall assume that $0 \in \Omega \setminus \Sigma$. (The proof for the case when $0 \in \sigma(T) \cap \Sigma$ follows by exactly the same argument.) Thus, we can find a finite sequence

$$t_1 = 0 < t'_1 < t_2 < t'_2 < \cdots < t_m < t'_m \leqslant t_{m+1} = 1,$$

such that $\gamma(t)$ belongs to some component Ω_j of $\Omega \setminus \Sigma$ for $t_j < t < t'_j$ and $\gamma(t)$ belongs to some component Σ_j of Σ for $t'_j \leqslant t \leqslant t_{j+1}$ ($j = 1, 2, \ldots, m$).

Let $\mu_j = \gamma(t_j)$, $\mu'_j = \gamma(t'_j)$ ($j = 1, 2, \ldots, m$), and let Σ_m denote the component of Σ containing $\gamma(1)$. (Beware: The Ω_j's are not necessarily distinct for different values of j, and the Σ_j's are not necessarily distinct for different values of j!)

A straightforward finite dimensional argument shows that there exists a decomposition $\mathscr{N}_0 = \bigoplus_{j=1}^m (\mathscr{L}_j \oplus \mathscr{L}'_j)$ of $\mathscr{N}_0$ such that

$$T_{11} = \begin{pmatrix} L_1 & & & & & & * \\ & L'_1 & & & & & \\ & & L_2 & & & & \\ & & & L'_2 & & & \\ & & & & \ddots & & \\ & & 0 & & & L_m & \\ & & & & & & L'_m \end{pmatrix}\begin{array}{l}\mathscr{L}_1\\ \mathscr{L}'_1\\ \mathscr{L}_2\\ \mathscr{L}'_2,\\ \vdots\\ \mathscr{L}_m\\ \mathscr{L}'_m\end{array}$$

where $\sigma(L_j) \subset \Omega_j$ and $\sigma(L'_j) \subset \Sigma_j$ for all $j = 1, 2, \ldots, m$.

First Step. Let γ_m be a Jordan arc included in Σ_m such that γ_m extends $\gamma\,|\,[t'_m, 1]$ to an arc including $\sigma(L'_m)$. Since $\gamma_m \subset \Sigma_m \subset \sigma_e(T)_{2\varepsilon} = \sigma_e(N)_{2\varepsilon}$, by using Lemma 3.2 we can find a finite dimensional subspace $\mathscr{S}_m$ of $\mathscr{R}_2$, reducing the diagonal normal operator N, a normal operator $N_m \in \mathscr{L}(\mathscr{S}_m)$ such that $\|N\,|\,\mathscr{S}_m - N_m\| \leqslant 2\varepsilon$, and a nilpotent operator $Q_m \in \mathscr{L}(\mathscr{L}'_m \oplus \mathscr{S}_m)$ such that $\|L'_m \oplus N_m - (\mu'_m + Q_m)\| < \varepsilon/m$. Thus

$$K'_m = T' - \begin{pmatrix} L_1 & & & & & & & & \\ & L'_1 & & & & & * & & \\ & & \ddots & & & & & & \\ & & & L'_{m-1} & & & & & \\ & & & & L_m & & & & \\ & & & & & \mu'_m + Q_m & & & \\ & & & & & & (A \oplus N) & & \\ & 0 & & & & & & N \mid \mathscr{S}_m{}^{\perp} & \\ & & & & & & & & T'_{22} \end{pmatrix} \begin{array}{l} \mathscr{L}_1 \\ \mathscr{L}'_1 \\ \vdots \\ \mathscr{L}'_{m-1} \\ \mathscr{L}_m \\ \mathscr{L}'_m \oplus \mathscr{S}_m \\ \mathscr{H}_1 \oplus \mathscr{R}_1 \\ \mathscr{R}_2 \ominus \mathscr{S}_m \\ \mathscr{H}_2 \end{array}$$

is the sum of two finite rank operators $K'_m(N)$ and $K'_m(Q)$, where $K'_m(N)$ is a normal operator, with $\|K'_m(N)\| \leqslant 3\varepsilon$, such that $\mathscr{S}_m$ reduces $K'_m(N)$ and $K'_m(N) \mid \mathscr{S}_m{}^{\perp} = 0$, and $\|K'_m(Q)\| < \varepsilon/m$.

Second Step. Now we analyze the operator

$$\begin{pmatrix} M_m & * \\ 0 & A \oplus N \end{pmatrix} \begin{array}{l} \mathscr{L}_m \oplus \mathscr{L}'_m \oplus \mathscr{S}_m \\ \mathscr{H}_1 \oplus \mathscr{R}_1 \end{array}, \quad \text{where } M_m = \begin{pmatrix} L_m & * \\ 0 & \mu'_m + Q_m \end{pmatrix} \begin{array}{l} \mathscr{L}_m \\ \mathscr{L}'_m \oplus \mathscr{S}_m \end{array}$$

satisfies $\sigma(M_m) \subset \Omega_m^-$. By Theorem 2.2, there exists a compact operator K_m, with $\|K_m\| < \varepsilon/m$, such that

$$A_m = \begin{pmatrix} M_n & * \\ 0 & A \oplus N \end{pmatrix} - K_m$$

is smooth; furthermore, a careful analysis of the proof of Theorem 2.2 (see, in particular [**1** and **11**, Corollary 3.42 and Lemma 3.46]) indicates that K_m can be chosen so that $\mathscr{L}_m \oplus \mathscr{L}'_m \oplus \mathscr{L}_m$ is invariant under A_m, and $\sigma(A_m \mid \mathscr{L}_m \oplus \mathscr{L}'_m \oplus \mathscr{S}_m) = \sigma(M_m)$. Moreover, since μ'_m and μ_m belong to *the same component* of $\rho^1_{\text{s-F}}(A_m) = \rho^1_{\text{s-F}}(A \oplus N)$, $\mathscr{L}_m \oplus \mathscr{L}'_m \oplus \mathscr{S}_m \subset \bigvee\{\ker(\mu_m - A_m)^k\}_{k=1}^{\infty}$.

Thus, if $\mathscr{N}_m = \bigvee\{\ker(\mu_m - A_m)^k\}_{k=1}^{n_m}$ for some n_m sufficiently large, then

$$\left\| P_{\mathscr{N}'_m} - P_{\mathscr{N}_m} P_{\mathscr{N}'_m} P_{\mathscr{N}_m} \right\| < \varepsilon/m,$$

where $\mathscr{N}'_m = \mathscr{L}_m \oplus \mathscr{L}'_m \oplus \mathscr{S}_m$.

Observe that $\sigma(A_m \mid \mathscr{N}_m) = \{\mu_m\} \subset \Sigma_{m-1}$. Let K''_m be the direct sum of K_m and the zero operator acting on the space $(\mathscr{H} \oplus \mathscr{R}_2) \ominus (\mathscr{N}'_m \oplus \mathscr{H}_1)$; K''_m is a compact operator ($\|K''_m\| < \varepsilon/m$) and

$$T_m = T' - \left(K'_m + K''_m\right)$$

$$= \begin{pmatrix} L_1 & & & & & & \\ & L'_1 & & & & * & \\ & & \ddots & & & & \\ & & & L_{m-1} & & & \\ & & & & L''_{m-1} & & \\ & 0 & & & & (A_m)_{\mathscr{N}_m^{\perp}} \oplus \left(N \mid \mathscr{S}_m{}^{\perp}\right) & \\ & & & & & & T'_{22} \end{pmatrix} \begin{array}{l} \mathscr{L}_1 \\ \mathscr{L}'_1 \\ \vdots \\ \mathscr{L}_{m-1} \\ \mathscr{L}'_{m-1} \oplus \mathscr{N}_m \\ [(\mathscr{H}_1 \oplus \mathscr{R}_1) \ominus \mathscr{N}_m] \oplus (\mathscr{R}_2 \ominus \mathscr{S}_m) \\ \mathscr{H}_2 \end{array}$$

where

$$L''_{m-1} = \begin{pmatrix} L'_{m-1} & * \\ 0 & A_m \mid \mathcal{N}_m \end{pmatrix} \begin{matrix} \mathcal{L}'_{m-1} \\ \mathcal{N}_m \end{matrix}$$

satisfies $\sigma(L''_{m-1}) \subset \Sigma_{m-1}$.

Inductive Step. Clearly, now we can repeat the argument of the First Step with the operator

$$\begin{pmatrix} L''_{m-1} & * \\ 0 & (A_m)_{\mathcal{N}_m^\perp} \oplus (N \mid \mathcal{S}_m^\perp) \end{pmatrix} \begin{matrix} \mathcal{L}'_{m-1} \oplus \mathcal{N}_m \\ [(\mathcal{H}_1 \oplus \mathcal{R}_1) \ominus \mathcal{N}_m] \oplus (\mathcal{R}_2 \ominus \mathcal{S}_m) \end{matrix};$$

then the Second Step can be repeated with the result of this operation, in order to obtain finite-rank operators $K'_{m-1}(N)$, $K'_{m-1}(Q_{m-1})$, and K''_{m-1}, where $K'_{m-1}(N)$ is reduced by a certain finite dimensional subspace $\mathcal{S}_{m-1}$ of $\mathcal{R}_1 \ominus \mathcal{S}_m$, $K'_{m-1}(N) \mid \mathcal{S}_{m-1}^\perp = 0$, $\|K'_{m-1}(N)\| \leqslant 3\varepsilon$, $\|K'_{m-1}(Q_{m-1})\| < \varepsilon/m$, and $\|K''_{m-1}\| < \varepsilon/m$. Moreover, we also obtain a finite dimensional invariant subspace $\mathcal{N}_{m-1}$ of the *smooth* operator

$$A_{m-1} = \begin{pmatrix} L''_{m-1} & * \\ 0 & (A_m)_{\mathcal{N}_m} \oplus (N \mid \mathcal{S}_m) \end{pmatrix} + \left(K'_{m-1} + K''_{m-1}\right)$$

($K'_{m-1} = K'_{m-1}(N) + K'_{m-1}(Q_{m-1})$) such that $\sigma(A_{m-1} \mid \mathcal{N}_{m-1}) = \{\mu_{m-1}\}$, and

$$\left\| P_{\mathcal{N}'_{m-1}} - P_{\mathcal{N}_{m-1}} P_{\mathcal{N}'_{m-1}} P_{\mathcal{N}_{m-1}} \right\| < 2\varepsilon/m,$$

where $\mathcal{N}'_{m-1} = \mathcal{L}_{m-1} \oplus \mathcal{L}'_{m-1} \oplus \mathcal{L}_m \oplus \mathcal{L}'_m \oplus \mathcal{S}_{m-1} \oplus \mathcal{S}_m$.

We repeat the same construction inductively. After m steps, we have constructed $3m$ finite rank operators, $K'_m(N)$, $K'_m(Q_m)$, K''_m, $K'_{m-1}(N)$, $K'_{m-1}(Q_{m-1})$, $K''_{m-1}, \ldots, K'_1(N)$, $K'_1(Q_1)$, K''_1, such that

$$\left\| \sum_{j=1}^{m} K'_j(N) \right\| = \max\left\{ \left\| K'_j(N) \right\| : 1 \leqslant j \leqslant m \right\} \leqslant 3\varepsilon$$

(recall that $K'_j(N)$ and $K'_i(N)$ only act nontrivially on orthogonal subspaces, for $j \neq i$),

$$\left\| \sum_{j=1}^{m} K'_j(Q_j) \right\| < m\left(\frac{\varepsilon}{m}\right) = \varepsilon, \quad \text{and} \quad \left\| \sum_{j=1}^{m} K''_j \right\| < m\left(\frac{\varepsilon}{m}\right) = \varepsilon,$$

and

$$T_1 = T' - \left[\sum_{j=1}^{m} \left(K'_j + K''_j \right) \right]$$

(where $K'_j = K'_j(N) + K'_j(Q_j)$) is smooth and admits a finite dimensional invariant subspace $\mathcal{N}_1$ such that $\sigma(T_1 \mid N_1) = \{0\}$, and

$$\left\| P_{\mathcal{N}'_1} - P_{\mathcal{N}_1} P_{\mathcal{N}'_1} P_{\mathcal{N}_1} \right\| < m(\varepsilon/m) = \varepsilon,$$

for a certain finite dimensional subspace $\mathcal{N}'_1$ including $\mathcal{N}_0 = \oplus_{j=1}^m (\mathcal{L}_j \oplus \mathcal{L}'_j)$. Clearly, the norm of the compact operator $K' = C_1 + C_2 + C_3 + \sum_{j=1}^m (K'_j + K''_j)$ does not exceed 8ε, and therefore $\|T_1\| < \|T\| + 8\varepsilon$.

By using Theorem 2.1, it is straightforward to check that $(T_1)_{\mathscr{H} \ominus \mathscr{N}_1}$ is also a smooth quasitriangular operator.

Rotation. Recall that

$$\sin \alpha_0 := \| P_{\mathscr{M}} - P_{\mathscr{N}_0} P_{\mathscr{M}} P_{\mathscr{N}_0} \| < \varepsilon < 1.$$

Thus, if $\mathscr{M}_0 = P_{\mathscr{N}_0}(\mathscr{M})$, then $\dim \mathscr{M}_0 = \dim \mathscr{M}$, and $\|P_{\mathscr{M}} - P_{\mathscr{M}_0}\| = \sin \alpha_0 < \varepsilon$. By using elementary trigonometry, we can find a unitary operator U_0 such that $\mathscr{M} + \mathscr{N}_0$ reduces U_0, $U_0 \mid (\mathscr{M} + \mathscr{N}_0)^{\perp} = 1$, $\|U_0 - 1\| = 2\sin(\alpha_0/2) < 2\varepsilon$, and $U_0 \mathscr{M}_0 = \mathscr{M}$, so that $U_0 \mathscr{N}_0 \supset \mathscr{M}$.

The same argument, applied to the inequality $\|P_{\mathscr{N}_0} - P_{\mathscr{N}_1} P_{\mathscr{N}_0} P_{\mathscr{N}_1}\| < \varepsilon$, yields a unitary operator U_1 of form "1 + finite rank", such that $\|U_1 - 1\| < 2\varepsilon$, and $U_1 \mathscr{N}_1 \supset \mathscr{N}_0$. It follows that $U_0 U_1 \mathscr{N}_1 \supset \mathscr{M}$, and $\|U_0 U_1 - 1\| < 4\varepsilon$.

Define $T_0 = U_0 U_1 T_1 (U_0 U_1)^*$; then

$$\begin{aligned} K_\varepsilon &= T \oplus N \oplus N - T_0 = K' + (T_1 - T_0) \in \mathscr{K}(\mathscr{H} \oplus \mathscr{R} \oplus \mathscr{R}), \\ \|K_\varepsilon\| &\leqslant \|K'\| + \|T_1 - T_0\| < 8\varepsilon + 8\|T_1\|\varepsilon \\ &< 8\varepsilon + 8(\|T\| + 8\varepsilon)\varepsilon < (100 + 10\|T\|)\varepsilon, \end{aligned}$$

and $T_0 = T \oplus N \oplus N - K_\varepsilon$ is a smooth quasitriangular operator, which admits a finite dimensional invariant subspace $\mathscr{N} = U_0 U_1 \mathscr{N}_1$ including $\mathscr{M}$, such that $\sigma(T_0 \mid \mathscr{N}) = \{0\}$, and $(T_0)_{\mathscr{N}^{\perp}}$ is also a smooth quasitriangular operator. □

LEMMA 3.4. *Let $T \in \mathscr{L}(\mathscr{H})$ be a smooth quasitriangular operator, let $N \in \mathscr{L}(\mathscr{R})$ be a diagonal normal operator of uniform infinite multiplicity such that $\sigma(N) = \sigma_e(T)$, and let $\mathscr{M}$ be a finite dimensional subspace of $\mathscr{R}$.*

Let $\Gamma = \{\lambda_j\}_{j=1}^{\infty}$ be a sequence of complex numbers such that

$$\lambda_h \in \sigma(T) \quad \textit{for all } h = 1, 2, \ldots, \quad \textit{and} \quad \operatorname{card}\{ j \colon \lambda_j \in \sigma \} = \aleph_0$$

for each nonempty clopen subset σ of $\sigma(T)$.

Given $p \geqslant 1$ and $\eta > 0$, there exist $K_\eta \in \mathscr{L}(\mathscr{H} \oplus \mathscr{R})$, with $\|K_\eta\| < \eta$, and a finite dimensional subspace $\mathscr{N}$ of $\mathscr{H} \oplus \mathscr{R}$ including $\mathscr{M}$, invariant under $T \oplus N - K_\eta$, such that

$$(T \oplus N - K_\eta) \mid \mathscr{N} = \begin{pmatrix} \lambda_1 & & & & & & & * \\ & \lambda_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_p & & & & \\ & & & & \lambda_{j_1} & & & \\ & & & & & \lambda_{j_2} & & \\ & & 0 & & & & \ddots & \\ & & & & & & & \lambda_{j_n} \end{pmatrix}$$

($p < j_1 < j_2 < \cdots < j_n$, $\dim \mathscr{N} = p + n$), and $T \oplus N - K_\eta$ and $(T \oplus N - K_\eta)_{\mathscr{N}^{\perp}}$ are smooth quasitriangular operators.

PROOF. Since $\lambda_1, \lambda_2, \ldots, \lambda_p \in \sigma(T) = \sigma_{\mathrm{lre}}(T) \cup \rho_{\text{s-F}}^{+}(T)$, each of these numbers is an approximate eigenvalue of any compact perturbation of T. By using this observation, the upper semicontinuity of the spectrum, and Theorem 2.2, it is not difficult to find $K_0 \in \mathscr{K}(\mathscr{H})$, with $\|K_0\| < \eta/4$, such that $T - K_0$ is a smooth quasitriangular operator with a p-dimensional invariant subspace $\mathscr{N}_0$, and

$$T - K_0 = \begin{array}{c} \left(\begin{array}{cccc|c} \lambda_1 & & & & \\ & \lambda_2 & & * & \\ & & \ddots & & \\ & 0 & & \lambda_p & \\ \hline & & & & T_0 \end{array}\right) \end{array} \begin{array}{l} \mathscr{N}_0 \\ \\ \mathscr{H} \ominus \mathscr{N}_0 \end{array}$$

(with respect to a suitable ONB of $\mathscr{N}_0$; T_0 is also smooth).

Let $\varepsilon = \eta/(400 + 40\|T_0\|)$. Clearly, $\sigma(T_0)_\varepsilon$ has only finitely many components $\sigma_1, \sigma_2, \ldots, \sigma_s$. Thus, we can write

$$T - K_0 = \left(\begin{array}{cccccccc} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & & & \\ & & \ddots & & & * & & \\ & & & \lambda_p & & & & \\ & & & & T_1 & & & \\ & & & & & T_2 & & \\ & & 0 & & & & \ddots & \\ & & & & & & & T_s \end{array}\right) \begin{array}{l} \\ \\ \mathscr{N}_0 \\ \\ \mathscr{H}_1, \\ \mathscr{H}_2 \\ \vdots \\ \mathscr{H}_s \end{array}$$

where $\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_s$ are defined so that $\mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \cdots \oplus \mathscr{H}_j$ is the Riesz spectral subspace of T_0 corresponding to the clopen subset $(\sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_j) \cap \sigma(T_0)$, $j = 1, 2, \ldots, s$.

Let $\mathscr{M}_j$ denote the orthogonal projection of $\mathscr{M}$ in $\mathscr{H}_j$, $j = 1, 2, \ldots, s$. By hypothesis, $N = \oplus_{j=1}^{s} N_j^{(s+2-j)}$, where $N_j \in \mathscr{L}(\mathscr{R}_j)$ is a diagonal normal operator of uniform infinite multiplicity such that $\sigma(N_j) \subset \sigma_j$, $j = 1, 2, \ldots, s$. Decompose

$$T \oplus N = T \oplus \left\{ \bigoplus_{j=1}^{s} N_j^{(2)} \right\} \oplus \left\{ \bigoplus_{j=1}^{s-1} N_j \right\} \oplus \left\{ \bigoplus_{j=1}^{s-2} N_j \right\} \oplus \cdots \oplus \{N_2 \oplus N_1\} \oplus N_1.$$

By Lemma 3.3, we can find $K_s \in \mathscr{L}(\mathscr{H}_s \oplus \mathscr{R}_s^{(2)})$, with $\|K_s\| < (100 + 10\|T_0\|)\varepsilon = \eta/4$, such that $T_s \oplus N_s^{(2)} - K_s$ is a smooth quasitriangular operator which admits a finite dimensional invariant subspace $\mathscr{N}_s$ including $\mathscr{M}_s$, and the spectrum of $(T_s \oplus N_s \oplus N_s - K_s) \mid \mathscr{N}_s$ is a singleton $\{\lambda_{(s)}\}$, where $\lambda_{(s)}$ is some limit point of Γ in σ_s.

Now, a new application of Lemma 3.3 produces an operator

$$K_{s-1} \in \mathscr{K}\left(\mathscr{H}_{s-1} \oplus \mathscr{R}_{s-1}^{(3)}\right),$$

with $\|K_{s-1}\| < \eta/4$, such that $T_{s-1} \oplus N_{s-1}^{(3)} - K_{s-1}$ is a smooth quasitriangular operator which admits a finite dimensional invariant subspace $\mathscr{N}_{s-1}$ including both $\mathscr{M}_{s-1}$ and the projection of $[(T_0 - K_0) \oplus \{\oplus_{j=1}^{s} N_j^{(2)}\}]\mathscr{N}_s$ in $\mathscr{H}_{s-1} \oplus \mathscr{R}_{s-1}^{(2)}$, and the spectrum of $(T_{s-1} \oplus N_{s-1}^{(3)} - K_{s-1}) \mid \mathcal{N}_{s-1}$ is a singleton $\{\lambda_{(s-1)}\}$, where $\lambda_{(s-1)}$ is some limit point of Γ in σ_{s-1}. Clearly, $\mathcal{N}_{s-1} \oplus \mathcal{N}_s$ is invariant under

$$\begin{pmatrix} T_{s-1} \oplus N_{s-1}^{(3)} - K_{s-1} & T_{s-1,s} \oplus 0 \\ 0 & T_s \oplus N_s - K_s \end{pmatrix},$$

where $T_{s-1,s}$ is the $(s-1,s)$-entry in the matrix representation of $T - K_0$; moreover, $\mathcal{N}_{s-1} \oplus \mathcal{N}_s$ includes $\mathcal{M}_{s-1} \oplus \mathcal{M}_s$.

By induction, we construct compact operators $K_s, K_{s-1}, K_{s-2}, \ldots, K_1$ ($K_j \in \mathcal{K}(\mathcal{H}_j \oplus \mathcal{R}_j^{(s+2-j)})$ and $\|K_j\| < \eta/4$ for all $j = 1, 2, \ldots, s$) and finite dimensional subspaces $\mathcal{N}_s, \mathcal{N}_{s-1}, \mathcal{N}_{s-2}, \ldots, \mathcal{N}_1$ ($\mathcal{N}_j \subset \mathcal{H}_j \oplus \mathcal{R}_j^{(s+2-j)}$, $j = 1, 2, \ldots, s$) in such a way that $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \cdots \oplus \mathcal{N}_{s-1} \oplus \mathcal{N}_S$ is invariant under

$$(T - K_0) \oplus N - \left(0_{\mathcal{N}_0} \oplus \left\{ \bigoplus_{j=1}^{s} K_j \right\}\right) = T \oplus N - K'_\eta,$$

where $K'_\eta = K_0 \oplus 0_{\mathcal{R}} + 0_{\mathcal{N}_0} \oplus \{\oplus_{j=1}^{s} K_j\} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{R})$ satisfies

$$\left\|K'_\eta\right\| \leqslant \|K_0\| + \max\left\{\|K_j\| : 1 \leqslant j \leqslant s\right\} < \eta/4 + \eta/4 = \eta/2;$$

moreover, $\mathcal{M} \subset \mathcal{N}_0 \oplus \{\oplus_{j=1}^{s} \mathcal{N}_j\} = \mathcal{N}$, $\sigma(T \oplus N - K'_\eta) = \sigma(T)$, and

$$\left(T \oplus N - K'_\eta\right) \mid \mathcal{N} = \begin{array}{c} \begin{pmatrix} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & * & & \\ & & \ddots & & & & & \\ & & & \lambda_p & & & & \\ & & & & T_{(1)} & & & \\ & & & & & T_{(2)} & & \\ & 0 & & & & & \ddots & \\ & & & & & & & T_{(s)} \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \mathcal{N}_0 \\ \\ \mathcal{N}_1, \\ \mathcal{N}_2 \\ \vdots \\ \mathcal{N}_s \end{array}$$

where $\sigma(T_{(j)}) = \{\lambda_{(j)}\}$, $j = 1, 2, \ldots, s$.

Since the $\lambda_{(j)}$'s are limit points of Γ, it is straightforward to find a finite-rank normal operator J_η, with $\|J_\eta\| < \eta/4$, such that $\mathcal{N} \ominus \mathcal{N}_0$ reduces J_η, $J_\eta \mid [\mathcal{N} \ominus \mathcal{N}_0]^\perp = 0$, and

$$\left[T \oplus N - \left(K'_\eta + J_\eta\right)\right] \mid \mathcal{N} = \begin{pmatrix} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & * & & \\ & & \ddots & & & & & \\ & & & \lambda_p & & & & \\ & & & & \lambda_{j_1} & & & \\ & & & & & \lambda_{j_2} & & \\ & 0 & & & & & \ddots & \\ & & & & & & & \lambda_{j_n} \end{pmatrix},$$

where $p < j_1 < j_2 < \cdots < j_n$ ($p + n = \dim \mathcal{N}$).

Finally, since $\sigma(T \oplus N - (K_\eta' + J_\eta)) = \sigma(T)$, we can use Theorem 2.2 *and its proof* (see comments in the Second Step of the proof of Lemma 3.3) in order to find $K_\eta'' \in \mathscr{K}(\mathscr{H} \oplus \mathscr{R})$, with $\|K_\eta''\| < \eta/4$, such that $T \oplus N - (K_\eta' + J_\eta + K_\eta'')$ is a smooth quasitriangular operator that leaves the finite dimensional subspace $\mathscr{N}$ invariant, the diagonal sequence of $[T \oplus N - (K_\eta' + J_\eta + K_\eta'')] | \mathscr{N}$ is equal to $\{\lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_n}\}$, and $[T \oplus N - (K_\eta' + J_\eta + K_\eta'')]_{\mathscr{N}^\perp}$ is also smooth.

Thus, the compact operator $K_\eta = K_\eta' + J_\eta + K_\eta''$ and the subspace $\mathscr{N}$ satisfy all our requirements. □

PROOF OF THEOREM 2.3. *Preparation*. Proceeding exactly as in the first part of the proof of [**15**, Proposition 4.4] (or [**2**, Proposition 13.11]), with the help of Voiculescu's theorem [**23**], after replacing (if necessary) T by an arbitarily small compact perturbation of this operator, we can directly assume that, instead of $T \in \mathscr{L}(\mathscr{H})$, we have to perturb the operator

$$L = T \oplus (A)_1 \oplus (A)_2 \oplus (A)_3 \oplus \cdots \in \mathscr{L}\left(\mathscr{H} \oplus \mathscr{H}_0^{(\infty)}\right),$$

where $(A)_k$ is a copy of an operator $A \in \mathscr{L}(\mathscr{H}_0)$ of the form

$$A = \begin{array}{c} \begin{pmatrix} N & F \\ 0 & G \end{pmatrix} \end{array}\begin{array}{l} \mathscr{H}_0(N) \\ \mathscr{H}_0(G) \end{array}.$$

Here N is a diagonal normal operator of uniform infinite multiplicity such that $\sigma(N) = \sigma_e(T)$, and G is a quasitriangular operator such that $\sigma(G) = \sigma_e(G) = \sigma_e(T)$. Thus, we have

$$L = \begin{pmatrix} T & & & & \\ & \begin{pmatrix} N & F \\ 0 & G \end{pmatrix} & & 0 & \\ & & \begin{pmatrix} N & F \\ 0 & G \end{pmatrix} & & \\ & & & \begin{pmatrix} N & F \\ 0 & G \end{pmatrix} & \\ & 0 & & & \ddots \end{pmatrix} \begin{array}{l} \mathscr{H} \\ \mathscr{H}_{0,1} \\ \mathscr{H}_{0,2} \\ \mathscr{H}_{0,3} \\ \vdots \end{array}$$

$$= \begin{pmatrix} \begin{pmatrix} T & 0 \\ 0 & N \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix} & & 0 & \\ & \begin{pmatrix} G & 0 \\ 0 & N \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix} & & \\ & & \begin{pmatrix} G & 0 \\ 0 & N \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix} & \\ & & & \begin{pmatrix} G & 0 \\ 0 & N \end{pmatrix} & \\ & 0 & & & \ddots \end{pmatrix} \begin{array}{l} \mathscr{H} \oplus \mathscr{H}_0(N)_1 \\ \mathscr{H}_0(G)_1 \oplus \mathscr{H}_0(N)_2 \\ \mathscr{H}_0(G)_2 \oplus \mathscr{H}_0(N)_3 \, . \\ \mathscr{H}_0(G)_3 \oplus \mathscr{H}_0(N)_4 \\ \vdots \end{array}$$

(This decomposition will play a very important role in the proof.)

First Step. Let $\{g_k\}_{k=1}^\infty$ be an ONB of $\mathscr{H} \oplus \mathscr{H}_0^{(\infty)}$ such that $g_1 \in \mathscr{H}$, and $g_k \in \mathscr{H} \oplus \{\oplus_{i=1}^{k-1} \mathscr{H}_{0,i}\}$ for all $k > 1$.

Let Δ be the (nonempty compact) set of limit points of the sequence Γ; then $\operatorname{dist}[\lambda_j, \Delta] \to 0$ $(j \to 0)$, and therefore $\operatorname{dist}[\lambda_j, \Delta] < \varepsilon/2$ for all $j > p_1$ $(\geqslant 1)$. By applying Lemma 3.4 to T, N, $\mathscr{M}_1 = \bigvee\{g_1\}$, Γ, p_1, and $\eta_1 = \varepsilon/2$, we can find a compact operator K_1, with $\|K_1\| < \varepsilon/2$, such that $\mathscr{H} \oplus \mathscr{H}_0(N)_1$ reduces K_1, $K_1 \,|\, [\mathscr{H} \oplus \mathscr{H}_0(N)_1]^\perp = 0$, $L - K_1$ is smooth, and $T \oplus N - K_1 \,|\, \mathscr{H} \oplus \mathscr{H}_0(N)_1$ admits a finite dimensional invariant subspace $\mathscr{N}_1$ including $\mathscr{M}_1$; moreover,

$$L_{1,1} := (L - K_1) \,|\, \mathscr{N}_1 = \left[T \oplus N - K_1 \,|\, \mathscr{H} \oplus \mathscr{H}_0(N)_1\right] \,|\, \mathscr{N}_1$$

admits an upper triangular matrix representation with diagonal sequence

$$d(L_{1,1}) = \Gamma_1 = \left\{\lambda_1, \lambda_2, \ldots, \lambda_{p_1}, \lambda^{(1)}_{j(1)}, \lambda^{(1)}_{j(2)}, \ldots, \lambda^{(1)}_{j(n_1)}\right\} \quad (p_1 + n_1 = \dim \mathscr{N}_1),$$

and this sequence is a finite subsequence of Γ.

Let $\Gamma_1' = \Gamma \setminus \Gamma_1$ be the subsequence of Γ obtained by eliminating all the elements in Γ_1. Recall that $L - K_1$ is smooth and quasitriangular. It readily follows from Theorem 2.1(x) that if $\lambda \in \rho_{\text{s-F}}(L - K_1)$, then $\bar{\lambda}$ cannot be an eigenvalue of $(L - K_1)^*$, and therefore $(L - K_1 - \lambda)^*$ is bounded below by a positive constant. Hence, there exists $\delta_1 > 0$ such that $(L - K_1 - \lambda)^*$ is bounded below by δ_1 for all λ in Γ_1 such that $L - \lambda$ is semi-Fredholm. It is easily seen that $(L - K_1 - \lambda) - B$ is also semi-Fredholm with the same index as $L - K_1 - \lambda$ and, moreover, that $\lambda \in \rho^r_{\text{s-F}}(L - K_1 - B)$, for all B in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H}_0^{(\infty)})$ such that $\|B\| < \delta_1$.

Second Step. Since $L - K_1$ is smooth, the spectrum of the operator

$$T_1 = \begin{pmatrix} T \oplus N - K_1 \,|\, \mathscr{H} \oplus \mathscr{H}_0(N)_1 & 0 \oplus F \\ 0 & G \end{pmatrix}$$

coincides with $\sigma((L - K_1)_{\mathscr{N}_1^\perp} = \sigma(T)$.

By definition of Δ, there exists p_2 such that $\lambda^{(1)}_{j(n_1)}$ precedes λ_{p_2} in the sequence Γ, and $\operatorname{dist}[\lambda_j, \Delta] < \varepsilon/4$ for all $j > p_2$. By applying Theorem 2.2 and Lemma 3.4 to T_1, N, $\mathscr{M}_2 :=$ orthogonal projection of $\bigvee\{g_2\}$ in $(\mathscr{H} \oplus \mathscr{H}_{0,1}) \ominus \mathscr{N}_1$ $(\dim \mathscr{M}_2 = 0$ or $1)$, Γ_1', p_2, and $\eta_2 = \min[\varepsilon/4, \delta_1/4]$, we can find a compact operator K_2, with $\|K_2\| < \varepsilon/4$, such that $[(\mathscr{H} \oplus \mathscr{H}_{0,1}) \ominus \mathscr{N}_1] \oplus \mathscr{H}_{0,2}$ reduces K_2, $K_2 \,|\, ([(\mathscr{H} \oplus \mathscr{H}_{0,1}) \ominus \mathscr{N}_1] \oplus \mathscr{H}_{0,2})^\perp = 0$, and $T_1 - K_2 \,|\, [(\mathscr{H} \oplus \mathscr{H}_{0,1}) \ominus \mathscr{N}_1] \oplus \mathscr{H}_{0,2}$ admits a finite dimensional invariant subspace $\mathscr{N}_2$ including $\mathscr{M}_2$; moreover,

$$L_{2,2} := (L - (K_1 + K_2)) \,|\, N_2 = \left[T_1 - K_2 \,|\, \left(\left[(\mathscr{H} \oplus \mathscr{H}_{0,1}) \ominus \mathscr{N}_1\right] \oplus \mathscr{H}_{0,2}\right)\right] \,|\, \mathscr{N}_2$$

admits an upper triangular matrix representation with diagonal sequence

$$d(L_{2,2}) = \left\{\lambda'^{(1)}_1, \lambda'^{(1)}_2, \ldots, \lambda'^{(1)}_{s_1 - 1}, \lambda_{p_2}, \lambda^{(2)}_{j(1)}, \lambda^{(2)}_{j(2)}, \ldots, \lambda^{(2)}_{j(n_2)}\right\} \quad (s_1 + n_2 = \dim \mathscr{N}_2),$$

where $\{\lambda'^{(1)}_1, \lambda'^{(1)}_2, \ldots, \lambda'^{(1)}_{s_1 - 1}, \lambda_{p_2}\}$ is the initial segment of Γ_1', and $d(L_{2,2})$ is a finite subsequence of Γ_1'. Furthermore, $[L - (K_1 + K_2)]_{\mathscr{N}_2^\perp}$ is smooth, and

$$L - (K_1 + K_2) = \begin{array}{c} \begin{pmatrix} L_{1,2} & * & * \\ 0 & L_{2,2} & * \\ 0 & 0 & T_2 \end{pmatrix} \begin{array}{l} \mathscr{N}_1 \\ \mathscr{N}_2 \\ \left[(\mathscr{H} \oplus \mathscr{H}_{0,1} \oplus \mathscr{H}_{0,2}) \ominus (\mathscr{N}_1 \oplus \mathscr{N}_2)\right] \end{array} \end{array}$$

$$\oplus (A)_3 \oplus (A)_4 \oplus (A)_5 \oplus \cdots,$$

where $L_{1,2}$ and $L_{1,1}$ are upper triangular with respect to the same ONB of $\mathcal{N}_1$ and $d(L_{1,2}) = d(L_{1,1})$, and $\|L_{1,2} - L_{1,1}\| < \eta_2$.

It is not difficult to check that

$$d\left(\begin{pmatrix} L_{1,2} & * \\ 0 & L_{2,2} \end{pmatrix}\right)$$

is a permutation of a finite subsequence Γ_2 of Γ (including the first $p_2 > 2$ in terms of Γ!), $\mathcal{N}_1 \oplus \mathcal{N}_2$ is invariant under $L - (K_1 + K_2)$ and includes $\bigvee\{g_1, g_2\}$, and $L - (K_1 + K_2)$ is a smooth operator because $\eta_2 \leqslant \delta_1/4$.

Define $\Gamma_2' = \Gamma \setminus \Gamma_2$. Proceeding as in the First Step, we can find δ_2, $0 < \delta_2 < \delta_1/8$, such that if $\lambda \in \Gamma_2$ is a regular point of the semi-Fredholm domain of $L - (K_1 + K_2)$, then $\lambda \in \rho^r_{\text{s-F}}(L - (K_1 + K_2) - B)$, for all B in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H}_0^{(\infty)})$ such that $\|B\| < \delta_2$.

Inductive step. By induction, we find compact operators $K_1, K_2, \ldots, K_k, \ldots$, and pairwise orthogonal finite dimensional subspaces $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_k$, such that $\|K_k\| < \eta_k \leqslant \varepsilon/2^k$ for all $k = 1, 2, \ldots$, and

$$L - \sum_{i=1}^{k} K_i = \begin{array}{l} \begin{pmatrix} L_{1,k} & & & & \\ & L_{2,k} & & * & \\ & & \ddots & & \\ & 0 & & L_{k,k} & \\ & & & & T_k \end{pmatrix} \begin{array}{l} \mathcal{N}_1 \\ \mathcal{N}_2 \\ \vdots \\ \mathcal{N}_k \\ \left[\left(\mathcal{H} \oplus \left\{\bigoplus_{i=1}^{k} \mathcal{H}_{0,i}\right\}\right) \ominus \left(\bigoplus_{i=1}^{k} \mathcal{N}_i\right)\right] \end{array} \\ \oplus (A)_{k+1} \oplus (A)_{k+2} \oplus (A)_{k+3} \oplus \cdots, \end{array}$$

where $\{L_{h,k}\}_{k=h}^{\infty}$ is a Cauchy sequence (in $\mathcal{L}(\mathcal{N}_h)$) of operators that are triangular with respect to a fixed ONB of $\mathcal{N}_h$, and $d(L_{h,k}) = d(L_{h,h})$ $(k \geqslant h)$, for each $h = 1, 2, \ldots$; moreover, $L - \sum_{i=1}^{k} K_i$ is a smooth quasitriangular operator, $\oplus_{i=1}^{k} \mathcal{N}_i \supset \bigvee\{g_i\}_{i=1}^{k}$, and $d((L - \sum_{i=1}^{k} K_i) \mid \oplus_{i=1}^{k} \mathcal{N}_i)$ is a permutation of a finite subsequence Γ_k of Γ, where $\Gamma_k \supset \{\lambda_1, \lambda_2, \ldots, \lambda_{p_k}\}$ for some $p_k > k$.

Furthermore, if $L_h = (\text{norm}) \lim_{k \to \infty} L_{h,k}$, and $K_\varepsilon = \sum_{k=1}^{\infty} K_k$, then $K_\varepsilon \in \mathcal{K}(\mathcal{H} \oplus \mathcal{H}_0^{(\infty)})$, $\|K_\varepsilon\| < \varepsilon$, $\mathcal{H} \oplus \mathcal{H}_0^{(\infty)} = \oplus_{k=1}^{\infty} \mathcal{N}_k$, and

$$(\text{norm}) \lim_{k \to \infty} \left(L - \sum_{i=1}^{k} K_i\right) = L - K_\varepsilon$$

$$= \begin{pmatrix} L_1 & & & & \\ & L_2 & & * & \\ & & \ddots & & \\ & & & L_k & \\ & 0 & & & \\ & & & & \ddots \end{pmatrix} \begin{array}{l} \mathcal{N}_1 \\ \mathcal{N}_2 \\ \vdots \\ \mathcal{N}_k \\ \vdots \end{array}$$

is a *smooth nontrivial operator* because the η_k's are small enough to guarantee that $\lambda_j \in \rho^r_{\text{s-F}}(L - K_\varepsilon)$ for all $j = 1, 2, \ldots$.

Clearly, $d(L - K_\varepsilon)$ is a permutation of Γ. But, by Lemma 2.6, this means that $d(L - K_\varepsilon) = \Gamma$ with respect to a suitable ONB of the space.

The proof of Theorem 2.3 is now complete. $\square$

CONJECTURE 3.5. Let γ be a Jordan arc joining 0 to 1, and let $M > 1$ and $\varepsilon > 0$ be given. There exists $m = m(\gamma, M, \varepsilon)$ such that if $F \in \mathscr{L}(\mathbf{C}^n)$ for some $n \geqslant m$, $\|F\| < M$ and $0, 1 \in \sigma(F) \subset \gamma$, then there exists $Q \in \mathscr{L}(\mathbf{C}^n)$ nilpotent such that $\|F - Q\| < \varepsilon$, provided the generalized eigenvalues of F (counted with multiplicity) are 'suitably spread' on γ.

An affirmative answer to the above conjecture will eliminate the need of Lemmas 3.2 and 3.3, and strongly simplify the proof of Theorem 2.3.

4. Characterization of strict $(-m)$-quasitriangularity. The intrinsic reason why the classes $(\mathrm{StrQT})_{-m}$ and $(\mathrm{QT})_{-m}$ $(m \geqslant 1)$ are 'tractable' and admit simple spectral characterizations (equivalences between the first and fifth statements in Theorems 1.2 and 1.2[a]) is, perhaps, that they are *invariant under similarity*. (Thus, the machinery developed in the monograph [**11**, **2**] can be applied to these classes.)

Although it is not strictly necessary for the proofs of Theorems 1.2 and 1.2[a], it will be shown, for the sake of completeness, that the approximating families of operators are also invariant under similarity. Indeed, we have the following

LEMMA 4.1. *Let $T \in \mathscr{L}(\mathscr{H})$; then*

(1) *T is $(-m)$-triangular if and only if*

(a$_1$) $\operatorname{nul} T^n \geqslant mn$, *for all $n = 1, 2, \ldots$; and*

(b$_1$) $\mathscr{H} = \bigvee_{n=1}^{\infty} \ker T^n$.

(2) *T is almost $(-m)$-triangular if and only if*

(a$_2$) *T admits an increasing chain $\{\mathscr{M}_k\}_{k=1}^{\infty}$ of finite dimensional invariant subspaces such that $\dim \mathscr{M}_{2k} \ominus \mathscr{M}_{2k-1} = m$ and $T_{\mathscr{M}_{2k} \ominus \mathscr{M}_{2k-1}} = 0$, for all $k = 1, 2, \ldots$; and*

(b$_2$) $\mathscr{H} = \bigvee_{k=1}^{\infty} \mathscr{M}_k$

PROOF. (1) The necessity of (a$_1$) and (b$_1$) follows by a straightforward computation of the matrices of the operators T^n, $n = 1, 2, \ldots$.

Assume that T satisfies (a$_1$) and (b$_1$); then there exists a family $\{\mathscr{R}_n\}_{n=1}^{\infty}$ of pairwise orthogonal subspaces of dimension m, such that $\mathscr{H} = \bigoplus_{n=1}^{\infty} \mathscr{R}_n$, and

$$T = \begin{array}{c} \begin{pmatrix} 0 & T_{12} & T_{13} & \cdots \\ & 0 & T_{23} & \cdots \\ & & 0 & \cdots \\ & & & \ddots \end{pmatrix} \end{array} \begin{array}{l} \mathscr{R}_1 \\ \mathscr{R}_2 \\ \mathscr{R}_3 \\ \vdots \end{array}.$$

Let $\{e_j\}_{j=1}^{\infty}$ be an ONB of $\mathscr{H}$ such that $\{e_1, e_2, \ldots, e_m\}$ is an ONB of $\mathscr{R}_1$, and $\{e_{(n-1)m+1}, e_{(n-1)m+2}, \ldots, e_{nm}\}$ is an ONB of $\mathscr{R}_n$, for each $n = 2, 3, \ldots$. It is easily seen that, if the basis $\{e_{(n-1)m+1}, e_{(n-1)m+2}, \ldots, e_{nm}\}$ $(n \geqslant 1)$ is cleverly chosen (by induction on n), then $T_{n,n+1}$ has an upper triangular $m \times m$ matrix, and therefore T is $(-m)$-triangular with respect to the basis $\{e_j\}_{j=1}^{\infty}$.

(2) Observe that (a_2)–(b_2) is essentially a reformulation of the definition of almost $(-m)$-triangularity. □

COROLLARY 4.2. *If $T \in \mathscr{L}(\mathscr{H})$ is (almost) $(-m)$-triangular, and $W \in \mathscr{G}(\mathscr{H})$, then WTW^{-1} is also (almost, respectively) $(-m)$-triangular.*

PROOF. Suppose that T is $(-m)$-triangular; then T satisfies (a_1) and (b_1) of Lemma 4.1(1), and these two conditions are obviously satisfied by WTW^{-1} for all W in $\mathscr{G}(\mathscr{H})$. By Lemma 4.1(1), WTW^{-1} is $(-m)$-triangular.

If we merely assume that T is almost $(-m)$-triangular, then it satisfies (a_2) and (b_2) of Lemma 4.1(2). Let $\mathscr{N}_k = W\mathscr{M}_k$ $(k = 1, 2, \ldots)$; then $\mathscr{N}_k$ is invariant under WTW^{-1}, $\mathscr{H} = W\mathscr{H} = \bigvee_{k=1}^{\infty}\mathscr{N}_k$, and a straightforward computation shows that $(WTW^{-1})_{\mathscr{N}_{2k}\ominus\mathscr{N}_{2k-1}} = 0$. By Lemma 4.1(2), WTW^{-1} is almost $(-m)$-triangular. □

Lemma 4.1(1) has a second consequence, which will play a very important role here.

COROLLARY 4.3. *Assume that $A \in \mathscr{L}(\mathscr{H}_1)$ is $(-p)$-triangular and $B \in \mathscr{L}(\mathscr{H}_2)$ is $(-q)$-triangular. Given $C \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$, and $\varepsilon > 0$, there exists $K_\varepsilon \in \mathscr{K}(\mathscr{H}_1 \oplus \mathscr{H}_2)$, with $\|K_\varepsilon\| < \varepsilon$, such that*

$$L = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\begin{matrix}\mathscr{H}_1 \\ \mathscr{H}_2\end{matrix} - K_\varepsilon$$

is $(-p - q)$-triangular.

PROOF. Let $\{e_j\}_{j=1}^{\infty}$ ($\{f_k\}_{k=1}^{\infty}$) be an ONB of $\mathscr{H}_1$ (of $\mathscr{H}_2$, resp.) such that A has a $(-p)$-triangular matrix (B has a $(-q)$-triangular matrix, resp.) with respect to this basis.

Clearly, we can find $K'_\varepsilon \in \mathscr{K}(\mathscr{H}_2, \mathscr{H}_1)$, with $\|K'_\varepsilon\| < \varepsilon/2$, such that $C_\varepsilon f_k \in \bigvee\{e_j\}_{j=1}^{n_k}$ (for some increasing sequence $\{n_k\}_{k=1}^{\infty}$), where $C_\varepsilon = C - K'_\varepsilon$. We can also find $K''_\varepsilon \in \mathscr{K}(\mathscr{H}_1)$, with $\|K''_\varepsilon\| < \varepsilon/2$, such that $A_\varepsilon = A - K''_\varepsilon$ is also $(-p)$-triangular (with respect to the same basis as A), and the $(j, j+p)$-entry of A_ε is different from zero for all $j = 1, 2, \ldots$.

Let

$$K_\varepsilon = \begin{pmatrix} K''_\varepsilon & K'_\varepsilon \\ 0 & 0 \end{pmatrix}\begin{matrix}\mathscr{H}_1 \\ \mathscr{H}_2\end{matrix};$$

clearly, $K_\varepsilon \in \mathscr{K}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ and $\|K_\varepsilon\| < \varepsilon$. (Indeed, K_ε can actually be chosen of arbitrarily small trace norm.) The operator

$$L = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\begin{matrix}\mathscr{H}_1 \\ \mathscr{H}_2\end{matrix} - K_\varepsilon = \begin{pmatrix} A_\varepsilon & C_\varepsilon \\ 0 & B \end{pmatrix}\begin{matrix}\mathscr{H}_1 \\ \mathscr{H}_2\end{matrix}$$

satisfies the following: for each $n \geqslant 1$,

$$L^n = \begin{pmatrix} A_\varepsilon^n & \sum_{i=1}^{n} A_\varepsilon^{n-i} C_\varepsilon B^{i-1} \\ 0 & B^n \end{pmatrix},$$

where A_ε^n is $(-np)$-triangular with respect to the ONB $\{e_j\}_{j=1}^{\infty}$, B^n is $(-nq)$-triangular with respect to the ONB $\{f_k\}_{k=1}^{\infty}$, and $(\sum_{i=1}^{n} A_\varepsilon^{n-i} C_\varepsilon B^{i-1}) f_k \in \bigvee\{e_j\}_{j=1}^{n_k}$ for each $k = 1, 2, \ldots$; moreover, the $(j, j+np)$-entry of A_ε^n is different from zero for all

$j = 1, 2, \ldots,$ and therefore ran A_ε^n contains all the finite linear combinations of the vectors $e_1, e_2, \ldots$.

In particular, the equation $A_\varepsilon^n e = -(\sum_{i=1}^n A_\varepsilon^{n-i} C_\varepsilon B^{i-1}) f_k$ has at least one solution, $e = g_k$, in the space $\bigvee\{e_j\}_{j=1}^{n_k}$ $(k = 1, 2, \ldots)$. It readily follows that

$$\ker L^n \supset \bigvee\{e_1, e_2, \ldots, e_{np}, g_1 + f_1, g_2 + f_2, \ldots, g_{nq} + f_{nq}\},$$

and therefore nul $L^N \geqslant np + nq = n(p+q)$, for all $n = 1, 2, \ldots$.

On the other hand, the above construction shows that $f_k \in \ker L^{n_k}$ for all $k = 1, 2, \ldots,$ so that

$$\bigvee_{n=1}^{\infty} \ker L^n \supset \bigvee\left(\{e_j\}_{j=1}^{\infty}\{f_k\}_{k=1}^{\infty}\right) = \mathscr{H}_1 \oplus \mathscr{H}_2.$$

By Lemma 4.1(1), L is a $(-p-q)$-triangular operator. □

In order to prove Theorem 1.2, we need another auxiliary result. The proof of Lemma 4.4 (below) follows by some ad hoc modifications of that of Lemma 3.1, and therefore it will be omitted.

LEMMA 4.4. *Let $T \in \mathscr{L}(\mathscr{H})$, and let $\Omega =$ interior Ω^- be a nonempty bounded open set such that $\partial\Omega \subset \sigma_{le}(T)$ and $\Omega^- \subset \sigma_{le}(T) \cup \{\lambda \in \mathbf{C}\colon \operatorname{nul}(\lambda - T) \geqslant n\}$ (for some $n \geqslant 1$). Given $\varepsilon > 0$, there exist $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon\| < \varepsilon$, and essentially normal operators $A_1, A_2, \ldots, A_n$, such that $\sigma(A_i) = \Omega^-$, $\sigma_e(A_i) = \partial\Omega$, $\operatorname{ind}(\lambda - A_i) = \operatorname{nul}(\lambda - A_i) = 1$ for all $\lambda \in \Omega$, and for all $i = 1, 2, \ldots, n$, and*

$$T - K_\varepsilon = \begin{pmatrix} \bigoplus_{i=1}^n A_i & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

satisfies

$$\min\operatorname{ind}(T - K_\varepsilon - \lambda)^k = \min\operatorname{ind}(T - \lambda)^k$$

for all $\lambda \in \rho_{s\text{-}F}(T)$, and for all $k = 1, 2, \ldots$.

PROOF OF THEOREM 1.2. First of all, observe that each of the following implications is either trivial or completely straightforward:

$$(\text{str-i})_{-m} \Rightarrow (\text{str-ii})_{-m} \text{ and } (\text{str-iii})_{-m};$$
$$(\text{str-vi})_{-m} \Rightarrow (\text{str-i})_{-m}, (\text{str-iv})_{-m} \text{ and } (\text{str-ix})_{-m},$$

because if $K \in \mathscr{K}(\mathscr{H})$, $\{R_j\}_{j=1}^{\infty}$, $\{Q_j\}_{j=1}^{\infty} \subset \mathscr{P}(\mathscr{H})$, and $R_j \to 1$ and $Q_j \to 1$ (strongly, as $j \to \infty$), then $\|K - R_j K Q_j\| \to 0$ $(j \to \infty)$ (and $(\text{str-xi})_{-m}$ (for some j, $1 \leqslant j \leqslant m$) $\Rightarrow (\text{str-v})_{-m}$, in the case when $m \geqslant 2$).

Furthermore, $(\text{str-i})_{-m}$ is easily seen to be equivalent to $(\text{str-ix})_{-m}$, and $(\text{str-ii})_{-m}$ $\Rightarrow (\text{str-i})_{-m}$ follows exactly as in the proof of the corresponding implication for strict m-quasitriangularity [**15**, §8, **2**, §13.5].

The distance formula can be obtained from W. B. Arveson's distance formula (from a given operator to a nest algebra [**3**]), as in [**15**, Corollary 7.2]. (We can also use the more direct formula due to S. C Powers; see [**20**].) Clearly, the validity of this formula implies the equivalence between statements $(\text{str-iii})_{-m}$ and $(\text{str-iv})_{-m}$.

If A is (m)-triangular, then there exists a unilateral shift S of multiplicity one such that $S^m A$ is triangular (with respect to the same ONB). By applying Proposition 1.1 to A, we infer that $\sigma_W(A) = \sigma_{\mathrm{lre}}(A) \cup \rho_{\mathrm{s\text{-}F}}^+(A)$ is a connected set containing the origin, and $\operatorname{ind}(\lambda - A) \geqslant 0$ for all $\lambda \in \rho_{\mathrm{s\text{-}F}}(A)$; by applying the same result to $S^m A$, we deduce that either A is not semi-Fredholm, or $\operatorname{ind} A \geqslant -\operatorname{ind} S^m = m$. Now, by using the upper semicontinuity of separate parts of the spectrum, and the stability properties of the semi-Fredholm index, we deduce that every (norm) limit of operators of the form $A + F$, where A is m-triangular and F has finite rank, has the same properties. Hence, $(\text{str-iv})_{-m} \Rightarrow (\text{str-v})_{-m}$.

If $m = 1$, then we can apply Corollary 2.5 to the special case when Γ is the sequence $\{0, 0, 0, \dots\}$, in order to obtain that $(\text{str-v})_{-1} \Rightarrow (\text{str-vi})_{-1}$.

We now prove that, in the case when $m \geqslant 2$, $(\text{str-v})_{-m}$ implies $(\text{str-vi})_{-m}$ and $(\text{str-xi})_{-m}$ (for all $j, 1 \leqslant j \leqslant m$). If T satisfies $(\text{str-v})_{-m}$ and T is a semi-Fredholm operator, then $\operatorname{ind} T \geqslant m$. By applying Lemma 4.4 with Ω equal to the component of interior $[\rho_{\mathrm{s\text{-}F}}^+(T)]^-$ including the origin, ε replaced by $\varepsilon/3$, and $n = m - 1$, we can find $K_\varepsilon' \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon'\| < \varepsilon/3$, such that

$$T - K_\varepsilon' = \begin{pmatrix} \bigoplus_{i=1}^{m-1} A_i & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

where A_i is essentially normal, $\sigma(A_i) = \Omega^-$, $\sigma_{\mathrm{e}}(A_i) = \partial\Omega$, and $\operatorname{ind}(\lambda - A_i) = \operatorname{nul}(\lambda - A_i) = 1$ for all all $\lambda \in \Omega$ $(i = 1, 2, \dots, m - 1)$; in this case, $\operatorname{ind}(\lambda - T_{22}) = \operatorname{ind}(\lambda - T) \geqslant 0$ for all $\lambda \in \rho_{\mathrm{s\text{-}F}}(T) \setminus \Omega$, and $\operatorname{ind}(\lambda - T) - (m - 1) \geqslant 1$ for all $\lambda \in \rho_{\mathrm{s\text{-}F}}(T) \cap \Omega$. If T satisfies $(\text{str-v})_{-m}$, but T is not semi-Fredholm, then we can proceed exactly as in the first lines of the proof of Corollary 2.5 in order to find $K_\varepsilon' \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon\| < \varepsilon/3$, such that

$$T - K_\varepsilon' \simeq T \oplus \left(\begin{pmatrix} 0 & F \\ 0 & G \end{pmatrix} \begin{matrix} \mathscr{H}_0(N) \\ \mathscr{H}_0(G) \end{matrix} \right)^{(m-1)} = \begin{pmatrix} 0^{(m-1)} & \begin{pmatrix} 0 & F^{(m-1)} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} T & 0 \\ 0 & G^{(m-1)} \end{pmatrix} \end{pmatrix} \begin{matrix} \mathscr{H}_0(N)^{(m-1)} \\ \mathscr{H} \\ \mathscr{H}_0(G)^{(m-1)} \end{matrix},$$

where $G \simeq G^{(\infty)}$ is quasitriangular, and $\sigma(G) = \sigma_{\mathrm{e}}(G) = \sigma_{\mathrm{e}}(T)$.

In either case, we have found $K_\varepsilon' \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon'\| < \varepsilon/3$, such that

$$T - K_\varepsilon' = \begin{pmatrix} \bigoplus_{i=1}^{m-1} A_i & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{matrix} \bigoplus_{i=1}^{m-1} \mathscr{H}_i \\ \mathscr{H}_0 \end{matrix},$$

where A_i is essentially normal and satisfies $(\text{str-v})_{-1}$ (for all $i = 1, 2, \dots, m - 1$), $\sigma(T_{22}) = \sigma(T)$, $\sigma_0(T_{22}) = \sigma_0(T)$, and T_{22} satisfies $(\text{str-v})_{-1}$.

By Corollaries 2.4 and 2.5, we can find compact operators $E_i \in \mathscr{K}(\mathscr{H}_i)$, with $\|E_i\| < \varepsilon/3$ $(i = 1, 2, \dots, m - 1)$, and $E_0 \in \mathscr{K}(\mathscr{H}_0)$, such that $A_i' = A_i - E_i$ is (-1)-triangular with $(j, j + 1)$-entries different from zero for all $j = 1, 2, \dots$ $(i = 1, 2, \dots, m - 1)$, and $T_{22}' = T_{22} - E_0$ is also (-1)-triangular; moreover, if $\sigma_0(T) = \varnothing$, then E_0 can be chosen with $\|E_0\| < \varepsilon/3$.

Let $E_\varepsilon = (\oplus_{i=1}^{m-1} E_i) \oplus E_0$; then

$$T - (K_\varepsilon' + E_\varepsilon) = \begin{array}{c} \\ \begin{pmatrix} \bigoplus_{i=1}^{m-1} A_i' & T_{12} \\ 0 & T_{22}' \end{pmatrix} \end{array} \begin{array}{l} \bigoplus_{i=1}^{m-1} \mathscr{H}_i \\ \mathscr{H}_0 \end{array}.$$

By an obvious inductive repetition of the argument of the proof of Corollary 4.3, now we can find $K_\varepsilon'' \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon''\| < \varepsilon/3$, of the form

$$K_\varepsilon'' = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \begin{array}{l} \bigoplus_{i=1}^{m-1} \mathscr{H}_i \\ \mathscr{H}_0 \end{array}$$

(i.e., the action of K_ε'' only affects T_{12}), such that the operator

$$A = T - (K_\varepsilon' + E_\varepsilon + K_\varepsilon'') = \begin{pmatrix} \bigoplus_{i=1}^{m-1} A_i' & T_{12}' \\ 0 & T_{22}' \end{pmatrix} \begin{array}{l} \bigoplus_{i=1}^{m-1} \mathscr{H}_i \\ \mathscr{H}_0 \end{array}$$

satisfies the conditions

$$A_{(\oplus_{i=j+1}^{m-1} \mathscr{H}_i) \oplus \mathscr{H}_0} = \begin{pmatrix} \bigoplus_{i=j+1}^{m-1} A_i' & P_{\oplus_{i=j+1}^{m-1} \mathscr{H}_i} T_{12}' \\ 0 & T_{22}' \end{pmatrix} \begin{array}{l} \bigoplus_{i=j+1}^{m-1} \mathscr{H}_i \\ \mathscr{H}_0 \end{array}$$

is $(j - m)$-triangular for all $j = 0, 1, 2, \ldots, m-2$.

Let $K_\varepsilon = K_\varepsilon' + E_\varepsilon + K_\varepsilon''$; clearly, K_ε is compact. Since $\oplus_{i=1}^{j} A_j'$ is obviously $(-j)$-triangular $(j = 1, 2, \ldots, m-1)$, by taking $C_j = K_\varepsilon$, we have proved that (str-xi)$_{-m}$ holds for all $j = 1, 2, \ldots, m-1$.

Since $T = A + K_\varepsilon$, and $\|K_\varepsilon\| < 2\varepsilon/3 + \max\{\varepsilon/3, \|E_0\|\}$, it readily follows that T satisfies (str-vi)$_{-m}$ and, moreover, that K_ε can be chosen so that $\|K_\varepsilon\| < \varepsilon$, provided $\sigma_0(T) = \varnothing$.

The proof of Theorem 1.2 is now complete. □

NOTE. The statement of Theorem 13.36 of [**2**, Chapter 13 (announcement of the results in Theorem 1.2) contains two minor errors: (1) The sentence "*T_j is unitarily equivalent to the direct sum of j copies of an essentially normal* (-1)-*triangular operator*" (statement (str-xi)$_{-m}$) must be changed to "*T_j is unitarily equivalent to the direct sum of j essentially normal* (-1)-*triangular operators*"; (2) The statement (str-xi)$_{-m,\varepsilon}$ is clearly not equivalent to (str-xi)$_{-m}$, unless $\sigma_0(T)$ is empty; therefore, statement (str-xi)$_{-m,\varepsilon}$ must be supressed from the theorem.

5. Characterization of $(-m)$-quasitriangularity. This section will be devoted to proving Theorem 1.2[a]. As in the case of Theorem 1.2, some implications are either trivial, or completely straightforward, or follow by essentially the same arguments as in the case of m-quasitriangularity [**11**, §9, **2** §13.6]:

(i)$_{-m}$ ⇔ (ii)$_{-m}$ ⇒ (iii)$_{-m}$;
(vi)$_{-m,\varepsilon}$ ⇒ (vi)$_{-m}$ ⇒ (i)$_{-m}$ and (ix)$_{-m}$;
(ix)$_{-m}$ ⇒ (xi)$_{-m}$;

(iv)$_{-m}$, (ix)$_{-m}$, or (xi)$_{-m}$ (for some $j, 1 \leqslant j \leqslant m$) $\Rightarrow$ (v)$_{-m}$ (use Proposition 1.1, the upper semicontinuity of the spectrum, and the stability properties of the index [**11**, Chapter 1, **19**]); and the validity of the distance formula, as well as the equivalence between (iii)$_{-m}$ and (iv)$_{-m}$, follow from some variation of W. B. Arveson's distance formula.

On the other hand, proceeding exactly as in the proof of Theorem 1.2(str-v)$_{-m} \Rightarrow$ (str-vi)$_{-m}$ and (str-xi)$_{-m}$ (for all j, $1 \leqslant j \leqslant m$) (with the help of Lemma 4.4 and Voiculescu's theorem) we can show that if T satisfies (v)$_{-m}$, then there exists $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon\| < \varepsilon$, such that

$$A_\varepsilon = T - K_\varepsilon = \begin{array}{c} \left(\begin{array}{ccccc} A_1 & & & & C_1 \\ & A_2 & & 0 & C_2 \\ & & \ddots & & \vdots \\ & 0 & & A_m & C_m \\ & & & & T_{22} \end{array}\right) \end{array} \begin{array}{c} \mathscr{H}_1 \\ \mathscr{H}_2 \\ \vdots \\ \mathscr{H}_m \\ \mathscr{H}_0 \end{array},$$

where A_i is an essentially normal (-1)-triangular operator with respect to an ONB $\{e_{i,j}\}_{j=1}^\infty$ of $\mathscr{H}_i$ $(i = 1, 2, \ldots, m)$, T_{22} is triangular with respect to an ONB $\{f_k\}_{k=1}^\infty$ of $\mathscr{H}_0$, and $C_k f_k \in \bigvee\{e_{i,j}\}_{j=1}^{n_{i,k}}$ (for some increasing sequence $\{n_{i,k}\}_{k=1}^\infty$, $i = 1, 2, \ldots, m$).

Thus, in order to prove that (v)$_{-m}$ implies (vi)$_{-m,\varepsilon}$ and (xi)$_{-m,\varepsilon}$, it suffices to show that $(A_\varepsilon)_{(\oplus_{i=j+1}^M \mathscr{H}_i)\oplus\mathscr{H}_0}$ is almost $(j - m)$-triangular for all $j = 0, 1, 2, \ldots, m - 1$. It will be shown that this is the case; clearly, we can reduce to the case when $j = 0$. Indeed, a slightly better result will actually be proved:

PROPOSITION 5.1. (Stringent form of $(-m)$-quasitriangularity.) *Let $T \in$ (QT), and assume that either $0 \in \sigma_{\mathrm{lre}}(T)$, or T is a semi-Fredholm operator with* $\operatorname{ind} T \geqslant m$. *Let $\{p_k\}_{k=1}^\infty$ be a strictly increasing sequence of natural numbers such that $p_{k+1} - p_k > m$ for all $k = 1, 2, \ldots$; then there exist $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon\| < \varepsilon$, and a maximal chain $\{P_j\}_{j=0}^\infty \subset \mathscr{P}(\mathscr{H})$, $P_j \uparrow 1$, such that*

$$(1 - P_j)(T - K_\varepsilon)P_{j+m} = 0 \quad \textit{for all } j \notin \bigcup_{k=1}^\infty \{p_k + 1, p_k + 2, \ldots, p_k + m\}$$

and

$$(1 - P_j)(T - K_\varepsilon)P_j = 0 \quad \textit{for all } j = 1, 2, \ldots.$$

PROOF. Choose K_ε as above, so that

$$A_\varepsilon = T - K_\varepsilon = \begin{pmatrix} \bigoplus_{i=1}^m A_i & C \\ 0 & T_{22} \end{pmatrix} \begin{array}{c} \bigoplus_{i=1}^m \mathscr{H}_i \\ \mathscr{H}_0 \end{array}, \quad \text{where } C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_m \end{pmatrix},$$

and the above conditions are fulfilled.

Let $n_k = \max\{n_{i,k}: 1 \leqslant i \leqslant m\}$. We can directly assume, without loss of generality, that the sequence $\{p_k\}_{k=1}^{\infty}$ is so sparse that $p_k > mn_k$ for all $k = 1, 2, \ldots$. Let P_h denote the orthogonal projection of $\mathscr{H}$ onto $\bigvee\{g_1, g_2, \ldots, g_h\}$, where $\{g_h\}_{h=1}^{\infty}$ is the ONB defined by

$$g_h = \begin{cases} e_{i,j} & \text{for } h = mj + i \leqslant p_1 + m, \\ f_1 & \text{for } h = p_1 + m + 1, \\ e_{i,j} & \text{for } p_1 + m + 2 \leqslant h = mj + i + 1 \leqslant p_2 + m, \\ f_2 & \text{for } h = p_2 + m + 1, \\ \cdots & \cdots \\ e_{i,j} & \text{for } p_{k-1} + m + 2 \leqslant h = mj + i + (k-1) \leqslant p_2 + m, \\ f_k & \text{for } h = p_k + m + 1, \\ \cdots & \cdots \end{cases}$$

(i.e., we 'interpolate' the f_k's very sparsely among the $e_{i,j}$'s).

Then $\{P_h\}_{h=0}^{\infty}$ $(P_0 = 0)$ is a maximal chain in $\mathscr{P}(\mathscr{H})$, $P_h \uparrow 1$,

$$(1 - P_h)A_\varepsilon P_{h+m} = 0 \quad \text{for all } h \notin \bigcup_{k=1} \{p_k + 1, p_k + 2, \ldots, p_k + m\}$$

and

$$(1 - P_h)A_\varepsilon P_h = 0 \quad \text{for all } h = 1, 2, \ldots. \qquad \square$$

This completes the proof of Theorem 1.2[a].

Proposition 5.1 makes no sense if the condition '$p_{k+1} - p_k > m$' is replaced by '$p_{k+1} - p_k > (m-1)$' for, in this case, the property

$$(1 - P_j)(T - K_\varepsilon)P_{j+m} = 0 \quad \text{for all } j \notin \bigcup_{k=1} \{p_k + 1, p_k + 2, \ldots, p_k + m\}$$

may be empty. (Take $p_k = (k-1)m$.) On the other hand, we have the following result. (Compare with [**15**, Conjecture 10.5].)

COROLLARY 5.2. (Relaxed form of strict $(-m)$-quasitriangularity.) *Let $T \in \mathscr{L}(\mathscr{H})$, and let $m \geqslant 2$. Suppose there exist a maximal chain $\{P_j\}_{j=1}^{\infty} \subset \mathscr{P}(\mathscr{H})$, $P_j \uparrow 1$, and an increasing sequence $\{p_k\}_{k=1}^{\infty}$ such that $p_{k+1} - p_k > m$ for all $k = 1, 2, \ldots,$ and*

$$\limsup_{j \to \infty} \left\{ \left\| (1 - P_j)TP_{j+m} \right\| : j \notin \bigcup_{k=1}^{\infty} \{p_k + 1, p_k + 2, \ldots, p_k + m - 1\} \right\} = 0.$$

Then $T \in (\mathrm{StrQT})_{-m}$.

PROOF. If T satisfies the above condition, then $T \in (\mathrm{QT})_{-m}$, and

$$\limsup_{j \to \infty} \left\| (1 - P_j)TP_{j+1} \right\| = 0,$$

so that $T \in (\mathrm{StrQT})_{-1}$.

Since T satisfies $(\text{str-v})_{-1}$ (Theorem 1.2), and $(\text{v})_{-m}$ (Theorem 1.2[a]), it is clear that T must also satisfy $(\text{str-v})_{-m}$. By Theorem 1.2, T is strictly $(-m)$-quasitriangular. $\square$

REMARK 5.3. As in the case of extended quasitriangularity, we can also consider the classes $(\mathrm{StrBQTStr})_{m,p} = (\mathrm{StrQT})_m \cap (\mathrm{StrQT})_p^*$, $(\mathrm{StrBQT})_{m,p} = (\mathrm{StrQT})_m \cap (\mathrm{QT})_p^*$, $(\mathrm{BQTStr})_{m,p} = (\mathrm{QT})_m \cap (\mathrm{StrQT})_p^*$, and, respectively, $(\mathrm{BQT})_{m,p}$, *for all possible integer values of m and p.*

It is not difficult to produce a few partial results by following the lines of §§4 and 5 of this article, together with the results of [**15**]. But these new classes present a large number of interesting open problems, of the type described in [**15**, §10, D and E].

6. Nest algebras. A *nest* $\mathbf{N}$ in $\mathscr{H}$ is a linearly ordered (by inclusion) family of subspaces containing $\{0\}$ and $\mathscr{H}$, such that the family $\{P_{\mathscr{M}}\colon \mathscr{M} \in \mathbf{N}\}$ is strongly closed in $\mathscr{L}(\mathscr{H})$. The order type of $\mathbf{N}$ is always that of a closed subset Δ of the real interval $[0,1]$ containing 0 and 1. A *gap* in N is a pair $\mathscr{g} = \{\mathscr{M}_-, \mathscr{M}_+\}$ of subspaces such that $\mathscr{M}_-$ is properly included in $\mathscr{M}_+$, and the conditions $\mathscr{M}_- \subset \mathscr{N} \subset \mathscr{M}_+$, $\mathscr{N} \in \mathbf{N}$ imply that either $\mathscr{N} = \mathscr{M}_-$, or $\mathscr{N} = \mathscr{M}_+$. (There is an obvious bijection between the gaps in $\mathbf{N}$ and the open intervals in $[0,1]\setminus\Delta$.) The cardinal number $\dim \mathscr{g} := \dim \mathscr{M}_+ \ominus \mathscr{M}_-$ is the *dimension* of the gap $\mathscr{g}$ ($1 \leqslant \dim \mathscr{g} \leqslant \aleph_0$).

Let Σ be a totally ordered denumerable set, and let $\{e_\nu\}_{\nu \in \Sigma}$ be an ONB of $\mathscr{H}$ ordered by Σ. The nest $\mathbf{N}_\Sigma$ generated by the subspaces $\mathscr{M}_{\mu-} = \bigvee\{e_\nu\}_{\nu<\mu}$ and $\mathscr{M}_{\mu+} = \bigvee\{e_\nu\}_{\nu\leqslant\mu}$ ($\mu \in \Sigma$) is order-isomorphic to an infinite totally disconnected closed subset Δ of $[0,1]$ containing 0 and 1; clearly, $\mathbf{N}_\Sigma$ has denumerably many gaps, and all its gaps are one dimensional. Conversely, given an infinite totally disconnected closed subset Δ of $[0,1]$, containing 0 and 1, there exist a totally ordered denumerable set $\Sigma \subset \Delta$ (Σ can be identified with the set of all 'first extreme points' of the segments in $[0,1]\setminus\Delta$) and a nest $\mathbf{N}_\Sigma$ of the above described type such that $\mathbf{N}_\Sigma$ is order-isomorphic to Δ. Furthermore, it is easily seen that $\mathbf{N}_\Sigma$ is unique, up to a unitary transformation of an ONB onto another one, and that two sets Δ_1 and Δ_2 (as above) produce the same nest (up to unitary equivalence) if and only if they are order-isomorphic. (In particular, they must be homeomorphic.)

A nest of this type can be properly called *generated by an orthonormal basis* (*gonb nest*, in short). The simplest example is the nest $\mathbf{N}_\omega$, corresponding to the case when Σ is the set of all natural numbers; $\mathbf{N}_{\mathbf{Z}}$ is the nest associated with an ONB $\{e_j\}_{j=-\infty}^{\infty}$, and the nest $\mathbf{N}_{\mathbf{Q}}$, corresponding to the case when $\Sigma = \mathbf{Q}$ is the set of all rational numbers, is order-isomorphic to the Cantor set, etc.

A word of caution must be said here: if Δ is an infinite totally disconnected closed subset of $[0,1]$ ($0, 1 \in \Delta$), but Δ *is not denumerable*, then there exists a nest $\mathbf{N}_\Delta$ order-isomorphic to Δ, all of whose gaps are one-dimensional, which is not a gonb nest! (To see this: Let $\mu = \mu_\Sigma + \mu_c$ be a probability measure on $[0,1]$, where Σ is the set of 'first extreme points' of $[0,1]\setminus\Delta$, μ_Σ is the atomic part of Σ, with an atom at each point of Σ, and μ_c is a nontrivial Borel measure supported by a perfect subset of Δ. Let H be the operator 'multiplication by t' on $L^2([0,1],\mu)$, and let $H = \int t\,dE_t$ be the spectral decomposition of H. Now define

$$\mathbf{N}_\Delta = \{\operatorname{ran} E([0,a)); \operatorname{ran} E([0,a])\colon a \in \Delta\}.)$$

However, a recent result of K. Davidson [**6**] guarantees that, given a nest $\mathbf{N}_\Delta$ as above, and ε ($0 < \varepsilon < 1$), there exist $U \in \mathscr{L}(\mathscr{H})$ unitary, $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon\| < \varepsilon$, and a gonb nest $\mathbf{N}_\Sigma$ such that

$$\mathbf{N}_\Delta = \{(U + K_\varepsilon)\mathscr{M}\colon \mathscr{M} \in \mathbf{N}_\Sigma\}.$$

(In particular, both $\mathbf{N}_\Delta$ and $\mathbf{N}_\Sigma$ are order-isomorphic to Δ.) This is good enough for our purposes.

The *nest algebra* associated with a nest $\mathbf{N}$ is defined by

$$\operatorname{alg}\mathbf{N} = \{T \in \mathscr{L}(\mathscr{H}) \colon T\mathscr{M} \subset \mathscr{M} \text{ for all } \mathscr{M} \text{ in } \mathbf{N}\}.$$

The reader is referred to [**3**, **6**, **18**, **21**] for general information about nests and their nest algebras. Following [**13**, **14**], we write

$$\hat{\mathbf{N}} = \{UAU^* + K \colon U \text{ is unitary}, A \in \operatorname{alg}\mathbf{N}, K \in \mathscr{K}(\mathscr{H})\},$$

and

$$\hat{\mathbf{N}}^0 = \{T \in \mathscr{L}(\mathscr{H}) \colon \text{Given } \varepsilon > 0, \text{ there exist } U_\varepsilon \text{ unitary}, A_\varepsilon \in \operatorname{alg}\mathbf{N},$$
$$\text{and } K_\varepsilon \text{ compact, with } \|K_\varepsilon\| < \varepsilon, \text{ such that } T = U_\varepsilon A_\varepsilon U_\varepsilon^* + K_\varepsilon\}.$$

In [**13**, **14**], the author completely characterized $\hat{\mathbf{N}}$ and $\hat{\mathbf{N}}^0$ (for all possible nests), thus solving a problem raised by W. B. Arveson:

(1) If $\mathbf{N} = \mathbf{N}_\Sigma$ for a *well-ordered* set Σ, then $\hat{\mathbf{N}} = \hat{\mathbf{N}}^0 = (\mathrm{QT})$ is the set of all quasitriangular operators. (In particular, $\hat{\mathbf{N}}_\omega = \hat{\mathbf{N}}_\omega^0 = (\mathrm{QT})$.)

(2) If $\mathbf{N} = \mathbf{N}_\Sigma$, where Σ is order-anti-isomorphic to a well-ordered set, then $\hat{\mathbf{N}} = \hat{\mathbf{N}}^0 = (\mathrm{QT})^* = \{T^* \colon T \in (\mathrm{QT})\}$. (3) In any other case, $\hat{\mathbf{N}} = \mathscr{L}(\mathscr{H})$; moreover, we have $\mathbf{N}^0 = \mathscr{L}(\mathscr{H})$ if and only if either $\mathbf{N}$ has infinitely many gaps, or at least one infinite dimensional gap. This holds, in particular, if $\mathbf{N}$ is order-isomorphic to a gonb nest, not of the types (1), or (2).

Let $\mathbf{N}$ be similar to a gonb nest; $\mathbf{N}$ has denumerably many one-dimensional gaps $\{\mathscr{G}_\nu\}_{\nu\in\Sigma}$. If $\mathscr{G}_\nu = \{\mathscr{M}_{\nu-}, \mathscr{M}_{\nu+}\}$, then it makes sense to talk of the diagonal entries of an operator A in $\operatorname{alg}\mathbf{N}$: observe that

$$A \mid \mathscr{M}_{\nu+} = \begin{pmatrix} A_\nu & * \\ 0 & a_\nu \end{pmatrix} \begin{matrix} \mathscr{M}_{\nu-} \\ \bigvee\{e_\nu\} \end{matrix},$$

where e_ν is a unit vector in $\mathscr{M}_{\nu+}$, orthogonal to $\mathscr{M}_{\nu-}$. We define a_ν as *the νth diagonal entry of A*.

In this setting, Theorem 2.3 and Corollary 2.5 can be easily reformulated as results about the structure of operators in $\hat{\mathbf{N}}_\omega^0$ and, respectively, $\hat{\mathbf{N}}_\omega$. What can be said for the other gonb nests? More precisely:

PROBLEM 6.1. Let $\mathbf{N}$ be a nest order-isomorphic to a totally disconnected closed subset Δ of $[0, 1]$, including 0 and 1. Assume that $\mathbf{N}$ has denumerable many gaps, totally ordered by an index set Σ, and that all the gaps are one dimensional. What kinds of relations can be expected between the totally ordered family $d(A) = \{a_\nu\}_{\nu\in\Sigma}$ of diagonal entries of A, and the different parts of the spectrum of A, for A in $\operatorname{alg}\mathbf{N}$?

Suppose that $T \in \hat{\mathbf{N}}$ ($\hat{\mathbf{N}}^0$), and $\Gamma = \{\lambda_\nu\}_{\nu\in\Sigma}$ is a family of complex numbers totally ordered by Σ. For which families Γ can we expect to find U unitary, A in $\operatorname{alg}\mathbf{N}$, and K compact (with $\|K\| < \varepsilon$, for a given $\varepsilon > 0$, resp.) such that $T = UAU^* + K$, and $d(A) = \Gamma$?

We are still very far from a complete answer to the above questions. The techniques introduced by C. Apostol, and the author (see [**1**, **2**, and **11**–**16**]) will certainly help, but the problem requires new arguments. For instance, the proof that 'most' gonb nests satisfy $\hat{\mathbf{N}}_\Sigma^0 = \mathscr{L}(\mathscr{H})$ actually uses a small portion of the nest: it suffices to assume that Σ includes both an increasing sequence and a decreasing sequence (and these two subsequences can be scattered anywhere in Σ [**14**]).

That is, no matter how complicated the order structure of Σ is, we can always have $\hat{\mathbf{N}}^0 = \mathscr{L}(\mathscr{H})$ provided Σ includes two 'nice' portions. On the other hand, if $T \in \mathscr{L}(\mathscr{H})$, and the *polynomial hull* $\sigma_e(T)\hat{}$ of the essential spectrum (i.e., the complement in $\mathbf{C}$ of the unbounded component of $\mathbf{C} \setminus \sigma_e(T)$) has two components, σ_1 and σ_2, then we cannot expect to find K compact such that $T - K$ is unitarily equivalent to an operator in $\mathbf{N}_\Sigma$ with a given diagonal family $\Gamma = \{\lambda_\nu\}_{\nu \in \Sigma}$, unless both,

$$\Sigma_1 = \{\nu \in \Sigma\colon \lambda_\nu \in \Omega_1\} \quad \text{and} \quad \Sigma_2 = \{\nu \in \Sigma\colon \lambda_\nu \in \Omega_2\}$$

(where Ω_j is a bounded open neighborhood of σ_j, $j = 1, 2$, and $\Omega_1 \cap \Omega_2 = \varnothing$), are 'sufficiently nice' (see Lemma 6.3 below).

A few results in the positive direction, and some examples, will illustrate the difficulties involved with Problem 6.1.

LEMMA 6.2. *Let $\mathbf{N}_\Sigma$ be a gonb nest, and let $A \in \operatorname{alg}\mathbf{N}_\Sigma$. Then $\sigma(A \mid \mathscr{M}) \cup \sigma(A_{\mathscr{M}^\perp}) \subset \sigma(A)\hat{}$ for all $\mathscr{M}$ in $\mathbf{N}_\Sigma$, and every diagonal entry of A belongs to $\sigma(A)\hat{}$.*

PROOF. The inclusions $\sigma(A \mid \mathscr{M}) \subset (A)\hat{}$ and $\sigma(A_{\mathscr{M}^\perp}) \subset \sigma(A)\hat{}$, for each subspace $\mathscr{M}$ invariant under A, follow from the main result of [**5**], applied to A, and to A^*.

Let $\{e_\nu\}_{\nu \in \Sigma}$ be the ONB generating $\mathbf{N}_\Sigma$, and let $\mathscr{M}_{\mu-} = \bigvee\{e_\nu\}_{\nu < \mu}$, and $M_{\mu+} = \bigvee\{e_\nu\}_{\nu \leqslant \mu}$ $(\mu \in \Sigma)$. Then $\mathscr{M}_{\mu-}$ and $\mathscr{M}_{\mu+}$ are invariant under $A \in \operatorname{alg}\mathbf{N}_\Sigma$, and $\sigma(A \mid \mathscr{M}_{\mu-}) \cup \sigma(A \mid \mathscr{M}_{\mu+}) \subset \sigma(A)\hat{}$.

Since

$$A \mid \mathscr{M}_{\mu+} = \begin{array}{c} \begin{pmatrix} A \mid \mathscr{M}_{\mu-} & * \\ 0 & a_\nu \end{pmatrix} \end{array} \begin{array}{l} \mathscr{M}_{\mu-} \\ \bigvee\{e_\mu\} \end{array},$$

it readily follows that $a_\nu \in \sigma(A)\hat{}$. $\square$

LEMMA 6.3. *Let $T \in \mathscr{L}(\mathscr{H})$, and let $\mathbf{N}_\Sigma$ be a gonb nest. Suppose that*

$$T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix} \begin{array}{l} \mathscr{H}_1 \\ \mathscr{H}_2 \end{array},$$

where $\mathscr{H}_1$ and $\mathscr{H}_2$ are infinite dimensional subspaces, and $\sigma(T_1)\hat{} \cap \sigma(T_2)\hat{} = \varnothing$.

Let $\Gamma = \{\lambda_\nu\}_{\nu \in \Sigma}$ be a totally ordered family of complex numbers such that $\lambda_\nu \in \sigma(T)\hat{}$ for all ν, and let $\Gamma_1 = \{\lambda_\nu\}_{\nu \in \Sigma_1}$ and $\Gamma_2 = \{\lambda_\nu\}_{\nu \in \Sigma_2}$, where $\Sigma_j = \{\nu \in \Sigma\colon \lambda_\nu \in \sigma(T_j)\hat{}\}$, $j = 1, 2$.

Suppose there exists $K \in \mathscr{K}(\mathscr{H})$ such that $T - K \cong A \in \operatorname{alg}\mathbf{N}_\Sigma$ and $d(A) = \Gamma$; then both Γ_1 and Γ_2 are infinite families,

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \begin{array}{l} \mathscr{H}_1' \\ \mathscr{H}_2' \end{array} \sim A_1 \oplus A_2,$$

where both, $\mathscr{H}_1'$ and $\mathscr{H}_2'$, are infinite dimensional subspaces, $\sigma(A_j) \subset \sigma(T_j)\hat{}$, A_j is similar to an operator in $\operatorname{alg}\mathbf{N}_{\Sigma_j}$, and $d(A_j) = \Gamma_j$, $j = 1, 2$.

Conversely, if Γ *is the disjoint union of two infinite subfamilies,* Γ_1 *and* Γ_2 *(as above), and some compact perturbation of* T_j *is similar to an operator* B_j *in* $\operatorname{alg}\mathbf{N}_{\Sigma_j}$ *such that* $d(B_j) = \Gamma_j$, $j = 1, 2$, *then there exists* $K \in \mathscr{K}(\mathscr{H})$ *such that* $T - K \cong A$ *for some* A *in* $\operatorname{alg}\mathbf{N}_\Sigma$ *such that* $d(A) = \Gamma$.

PROOF. Assume that $T = UAU^* + K$, where $K \in \mathscr{K}(\mathscr{H})$, U is unitary, and $A \in \operatorname{alg}\mathbf{N}$ with $d(A) = \Gamma$. It follows from Lemma 6.2 that $\sigma(A) \subset \sigma(T)\hat{} = \sigma(T_1)\hat{} \cup \sigma(T_2)\hat{}$ (disjoint union); therefore,

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \begin{matrix} H_1' \\ H_2' \end{matrix},$$

where $\mathscr{H}_1' = \mathscr{H}(\sigma(T_1)\hat{}\,; A)$ and $\mathscr{H}_2' = \mathscr{H} \ominus \mathscr{H}_1'$.

Clearly, $A \sim A_1 \oplus A_2$ [**12**, Chapter 3, **19**], so that $A_1 \oplus A_2$ admits a nest of invariant subspaces order-isomorphic to $\mathbf{N}_\Sigma$, all of whose gaps are one dimensional. Since every invariant subspace of $A_1 \oplus A_2$ is the orthogonal direct sum of a subspace of $\mathscr{H}_1'$ invariant under A_1, and a subspace of $\mathscr{H}_2'$ invariant under A_2 [**17**], it is not difficult to deduce that Γ_j is an infinite family, A_j admits a nest order-isomorphic to $\mathbf{N}_{\Sigma_j}$, all of whose gaps are one dimensional, and $d(A_j) = \Gamma_j$, $j = 1, 2$. By the above-mentioned result of K. Davidson [**6**], A_j is actually similar to an operator in $\operatorname{alg}\mathbf{N}_{\Sigma_j}$, $j = 1, 2$.

Conversely, if $T_j = U_j B_j U_j^* + K_j$, where U_j is unitary, $K_j \in \mathscr{K}(\mathscr{H}_j)$, $B_j \in \operatorname{alg}\mathbf{N}_{\Sigma j}$, and $d(B_j) = \Gamma_j$ (an infinite subfamily of Σ), $j = 1, 2$, then

$$T_1 \oplus T_2 = (U_1 \oplus U_2)(B_1 \oplus B_2)(U_1 \oplus U_2)^* + K_1 \oplus K_2,$$

and every operator similar to $B_1 \oplus B_2$ has a nest of invariant subspaces order-isomorphic to $\mathbf{N}_\Sigma$, all of whose gaps are one dimensional.

Since $K_1 \oplus K_2 \in \mathscr{K}(\mathscr{H})$, and $T \sim T_1 \oplus T_2$, it follows that some compact perturbation $A' = T - K'$ of T admits a nest of invariant subspaces of that form; moreover, $d(A') = \Gamma$. Finally, by using K. Davidson's result [**6**], we can find W invertible of the form $U + C$ (where U is unitary and C is compact) such that $A = (U + C)^{-1} A'(U + C) \in \operatorname{alg}\mathbf{N}_\Sigma$, and $d(A) = d(A') = \Gamma$.

Since $K = T - UAU^* = K' + [(U + C)A(U + C)^{-1} - UAU^*] \in \mathscr{K}(\mathscr{H})$, the proof is complete. □

The result of Lemma 6.2 is the best possible, in general. For instance, if $\Gamma = \mathbf{Z}$, we can actually have all the diagonal entries of A in $\sigma(A)\hat{}\setminus\sigma(A)$.

PROPOSITION 6.4. *Let* $T \in \mathscr{L}(\mathscr{H})$, *and let* $\Gamma = \{\lambda_n\}_{n=-\infty}^{\infty}$ *be a two-sided sequence of complex numbers such that*

(i) *all the limit points of* Γ *belong to* $\sigma_e(T)\hat{}$; *and*

(ii) $\operatorname{card}\{n < 0: \lambda_n \in \Omega\} = \operatorname{card}\{n > 0: \lambda_n \in \Omega\} = \aleph_0$ *for each open set* Ω *such that* $\Omega \cap \sigma(T)\hat{} \neq \varnothing$, *but* $\partial\Omega \cap \sigma(T)\hat{} = \varnothing$.

Then there exist $K \in \mathscr{K}(\mathscr{H})$, U *unitary, and* $A \in \operatorname{alg}\mathbf{N}_\Sigma$ *such that*

(1) $T = UAU^* + K$; *and*

(2) $d(A) = \Gamma$.

Furthermore, if Γ *also satisfies*

(iii) $\lambda_n \in \sigma(T)^\wedge$ *for all* $n \in \mathbf{Z}$; *and*

(iv) $\operatorname{card}\{n: \lambda_n = \lambda\} = \dim \mathscr{H}(\lambda; T)$ *for all* $\lambda \in \sigma_0(T) \setminus \sigma_e(T)^\wedge$; *then, given* $\varepsilon > 0$, K *can be chosen so that* $\|K\| < \varepsilon$,

(3) $\sigma_0(A) = \sigma_0(T)$;

(4) *if* $\Delta_\varepsilon = \{\lambda \in \sigma_0(T): \operatorname{dist}[\lambda, \sigma_e(T)] > \varepsilon\}$, *then*

$$\mathscr{H}(\Delta_\varepsilon; UAU^*) = \mathscr{H}(\Delta_\varepsilon; T) \quad \textit{and} \quad UAU^* \mid \mathscr{H}(\Delta_\varepsilon; UAU^*) = T \mid \mathscr{H}(\Delta_\varepsilon; T);$$

and

(5) *if* $\lambda \in \sigma_0(T)$, *then* $\dim \mathscr{H}(\lambda; A) = \dim \mathscr{H}(\lambda; T)$, *and* $A \mid \mathscr{H}(\lambda; A) \sim T \mid \mathscr{H}(\lambda; T)$.

PROOF. By proceeding as in the proof of [**13**, Theorem 1, **2**, §A3.2], given $\varepsilon > 0$, we can find $K_1 \in \mathscr{K}(\mathscr{H})$, with $\|K_1\| < \varepsilon/4$, such that

$$T - K_1 \cong T \oplus A, \quad A = \begin{pmatrix} A_+ & * \\ 0 & A_- \end{pmatrix}$$

where $\sigma(A_+) = \sigma_{re}(A_+) = \sigma(A_-) = \sigma_{le}(A_-) = \sigma_e(T)^\wedge$.

We can write

$$T - K_1 \cong \begin{pmatrix} T_+ & * & * \\ 0 & T_0 & * \\ 0 & 0 & T_- \end{pmatrix},$$

where

$$\begin{pmatrix} T_+ & * \\ 0 & T_0 \end{pmatrix} = T \oplus A_+,$$

T_0 is the compression of T to $\mathscr{H}([\sigma(T) \setminus \Delta_{\varepsilon/4}]^\wedge; T)^\perp$, and $T_- = A_-$ ($\sigma(T \oplus A_+) = \sigma_0(T) \cup \sigma_e(T)^\wedge$, $\sigma_e(T_+) = \sigma_{re}(T_+) = \sigma_{re}(A_+) = \sigma_e(T)^\wedge$, and $\sigma(T_+) \subset [\sigma_e(T_+)^\wedge]_{\varepsilon/4}$).

By standard arguments (as in the first part of the proof of Corollary 2.4, in §2), we can rapidly reduce the proof of both statements to show that if Γ satisfies (ii) and

(i′) $\lambda_n \in \sigma_e(T)^\wedge$ for all $n \in \mathbf{Z}$,

and

$$B = \begin{pmatrix} B_+ & B_{12} \\ 0 & B_- \end{pmatrix} \begin{matrix} \mathscr{H}_+ \\ \mathscr{H}_- \end{matrix},$$

where $\sigma(B_+) = \sigma_{re}(B_+) = \sigma(B_-) = \sigma_{le}(B_-) = \sigma_e(T)^\wedge$, then there exists $K_2 \in \mathscr{K}(\mathscr{H})$, with $\|K_2\| < \varepsilon/4$, such that $B - K_2 \cong B' \in \operatorname{alg} \mathbf{N_Z}$, with $d(B') = \Gamma$. But this is an immediate consequence of Theorem 2.3:

(a) Apply the theorem to the quasitriangular operator B_- ($\sigma(B_-) = \sigma_{le}(B_-) = \sigma_e(T)^\wedge$) and the sequence $\Gamma_- = \{\lambda_n\}_{n=0}^{\infty}$, in order to find $K_{2-} \in \mathscr{K}(\mathscr{H}_-)$ with $\|K_{2-}\| < \varepsilon/4$, such that $B'_- = B_- - K_{2-}$ is triangular with $d(B'_-) = \Gamma_-$.

(b) Apply the theorem to the quasitriangular operator B'_+ ($\sigma(B_+^*) = \sigma_{le}(B_+^*) = \sigma_{re}(B_+)^* = [\sigma_e(T)^\wedge]^*$) and the sequence $\Gamma_+ = \{\bar\lambda_{-n}\}_{n=1}^{\infty}$, in order to find $K_{2+}^* \in \mathscr{K}(\mathscr{H}_+)$, with $\|K_{2+}^*\| < \varepsilon/4$, such that $(B'_+)^* = (B_+ - K_{2+})^*$ is triangular with $d((B'_+)^*) = \Gamma_+$.

(Conditions (i′) and (ii) guarantee that both constructions are possible.)

Now define $K_2 = K_{2+} \oplus K_{2-}$; then $K_2 \in \mathscr{K}(\mathscr{H})$, $\|K_2\| < \varepsilon/4$, and

$$B - K_2 = \begin{pmatrix} B'_+ & B_{12} \\ 0 & B'_- \end{pmatrix} \begin{matrix} \mathscr{H}_+ \\ \mathscr{H}_- \end{matrix} \simeq B' \in \operatorname{alg} \mathbf{N}_{\mathbf{Z}},$$

where $d(B') = \Gamma$. $\square$

EXAMPLE 6.5. Let $L \in \mathscr{L}(\mathscr{H}_-)$ ($R \in \mathscr{L}(\mathscr{H}_+)$), and assume that $\sigma(L) = \sigma_W(L)$ ($\sigma(R) = \sigma_W(R)$, resp.) is a connected subset of the open left (right, resp.) halfplane such that $\sigma(L)\hat{}$ ($\sigma(R)\hat{}$, resp.) contains the point -1 ($+1$, resp.). Let $T = L \oplus R$, and let $\Gamma = \{\lambda_n\}_{n=-\infty}^{\infty}$, where $\lambda_n = -1$, or $\lambda_n = 1$ for all $n \in \mathbf{Z}$.

Under what conditions on T and on Γ can we guarantee that, for each $\varepsilon > 0$, there exists $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon\| < \varepsilon$, such that $T - K_\varepsilon \cong A \in \operatorname{alg} \mathbf{N}_{\mathbf{Z}}$, with $d(A) = \Gamma$?

The conditions $\operatorname{card}\{n\colon \lambda_n = 1\} = \operatorname{card}\{n\colon \lambda_n = 1\} = \aleph_0$ are *necessary* in all cases, and (by Proposition 6.4), the conditions

$$\operatorname{card}\{n < 0\colon \lambda_n = -1\} = \operatorname{card}\{n < 0\colon \lambda_n = 1\}$$
$$= \operatorname{card}\{n > 0\colon \lambda_n = -1\} = \operatorname{card}\{n > 0\colon \lambda_n = 1\} = \aleph_0$$

are *sufficient* in all cases.

Assume that $\lambda_{-n} = -1$ and $\lambda_n = 1$ for all $n \geqslant n_0$; then the answer is affirmative if and only if L^*, $R \in$ (QT), $-1 \in \sigma(L)$, and $1 \in \sigma(R)$.

Assume that $\lambda_n = 1$ for all $n \geqslant n_0$, but $\operatorname{card}\{n < 0\colon \lambda_n = -1\} = \operatorname{card}\{n < 0\colon \lambda_n = 1\} = \aleph_0$; then the answer is affirmative if and only if $L^* \in$ (QT) and $-1 \in \sigma(L)$.

The other cases can be similarly analyzed. Even an example as simple as this one produces seven different cases!

Another interesting case is the one corresponding to a well-ordered family Σ. As we have already observed, $\hat{\mathbf{N}}_\Sigma^0 = \hat{\mathbf{N}}_\Sigma$ always coincides with (QT), the class of all quasitriangular operators [**14**], and Corollaries 2.4 and 2.5 completely solve Problem 6.1 for the case when Σ is the set of all natural numbers. The following analogue to Proposition 1.1 can be deduced from [**13**, Lemma 7 and **14**]:

PROPOSITION 6.6. *Let Σ be a denumerable well-ordered family of indices, and let $A \in \operatorname{alg} \mathbf{N}_\Sigma$ ($d(A) = \{a_\nu\}_{\nu \in \Sigma}$); then*

(i) $d(A) \subset \sigma(A) = \sigma_l(A) = \sigma_{\mathrm{lre}}(A) \cup \rho_{\mathrm{s\text{-}F}}(A) \cup \sigma_0(A)$.

(ii) *Every nonempty clopen subset of $\sigma(A)$ intersects $d(A)$, and every component of $\sigma(A)$ intersects $d(A)^-$; furthermore, if σ is a clopen subset of $\sigma(A)$ such that $\sigma \cap \sigma_e(A) \neq \varnothing$, then* $\operatorname{card}\{\nu \in \Sigma\colon a_\nu \in \sigma\} = \aleph_0$.

(iii) *Every isolated point of $\sigma(A)$ belongs to $d(A) \cap \sigma_p(A)$; moreover, if $\lambda \in \sigma_0(A)$, then* $\operatorname{card}\{\nu \in \Sigma\colon a_\nu = \lambda\} = \dim \mathscr{H}(\lambda; A)$.

(iv) *If* $\ker(\lambda - A)^* \neq \{0\}$, *then $\lambda \in d(A)$, so that $\sigma_p(A^*)$ is an at most denumerable subset of $d(A)^* = \{\bar{\lambda}\colon \lambda \in d(A)\}$.*

(v) *If $\lambda - A$ is semi-Fredholm, then* $\operatorname{ind}(\lambda - A) \geqslant 0$.

(vi) *Let $1 \leqslant n < m < \infty$, and assume that $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3 + \cdots + \Sigma_m$ (ordinal sum: each element of Σ_h precedes each element of Σ_{h+1}, $h = 1, 2, \ldots, m-1$); then the families $\{a_\nu\colon \nu \in \Sigma_h\}$ have a limit point in the same component of $\rho_{\mathrm{s\text{-}F}}^n(A)$ for at most n distinct indices h, $1 \leqslant h \leqslant m$.*

(vii) *If* Σ *has a last element*, β, *and* α *is the last limit ordinal in* Σ (*so that* $\beta = \alpha + p - 1$ *for some* p, $1 \leqslant p < \infty$), *then* $\{a_\alpha, a_{\alpha+1}, a_{\alpha+2}, \ldots, a_{\alpha+p-2}, a_{\alpha+p-1} = a_\beta\} \subset \sigma_p(A^*)^*$. *In particular, if* $a_{\alpha+j-1} \in \rho_{\text{s-F}}(A)$ (*for some* j, $1 \leqslant j \leqslant p$), *then* $a_{\alpha+j-1}$ *is a singular point of the semi-Fredholm domain of* A.

PROOF. Statements (i) through (v) were proved in the above-mentioned references, and (vii) follows immediately from Theorem 2.1 and a simple analysis of the matrix of A^*.

(vi) If $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3 + \cdots + \Sigma_m$, then

$$A = \begin{pmatrix} A_1 & & & & \\ & A_2 & & * & \\ & & A_3 & & \\ & 0 & & \ddots & \\ & & & & A_m \end{pmatrix}, \quad d(A_j) = \{a_\nu \colon \nu \in \Sigma_h\}, \qquad h = 1, 2, \ldots, m.$$

Let Ω be a component of $\rho^n_{\text{s-F}}(A)$. If $d(A_h)$ has a limit point in Ω, then it follows from (i) that $\operatorname{ind}(\lambda - A_h) > 0$ for all λ in Ω. Since

$$n = \operatorname{ind}(\lambda - A) = \sum_{h=1}^{m} \operatorname{ind}(\lambda - A_h) \qquad (\lambda \in \Omega),$$

it readily follows that $\{a_\nu \colon \nu \in \Sigma_h\}$ cannot have a limit point in Ω for more than n distinct indices h, $1 \leqslant h \leqslant m$. □

Corollaries 2.4 and 2.5 roughly say that the only restrictions that we have to put on the sequence Γ in order to solve Problem 6.1 for the case $\Sigma = N$, are the natural ones that we can derive from Proposition 1.1. It is not clear whether the natural restrictions that we can derive from Lemma 6.6 will provide a similar answer for the case of a more general well-ordered family Σ.

WORK IN PROGRESS. In his article **[25]**, D. R. Larson negatively answered two important questions due to R. V. Kadison and I. M. Singer and, respectively, to J. R. Ringrose, concerning nests of subspaces included in the invariant subspace lattice of a given operator. In this important article, Larson introduced the ideal $\mathscr{R}^\infty_{\mathscr{N}}$ of a nest algebra $\operatorname{alg}\mathscr{N}$: $\mathscr{R}^\infty_{\mathscr{N}}$ is the class of all operators A in $\operatorname{alg}\mathscr{N}$ with the property that, given $\varepsilon > 0$, there exists a (perhaps infinite) family $\{\Delta_n\}$ of pairwise disjoint (open, closed, or semiopen) intervals of $[0,1]$ such that $\bigvee_n E(\Delta_n)\mathscr{H} = \mathscr{H}$ and $\|E(\Delta_n)AE(\Delta_n)\| < \varepsilon$ for all n (where $E(\cdot)$ is the spectral measure associated to $\mathscr{N}$).

Recently, D. R. Larson and the author have obtained spectral characterizations of the sets

$$\hat{\mathscr{R}}^\infty_{\mathscr{N}} = \{UAU^* + K \colon U \text{ is unitary, } A \in \mathscr{R}^\infty_{\mathscr{N}},\ K \in \mathscr{K}(\mathscr{H})\}$$

for all possible nests $\mathscr{N}$. (That is, the analogues for $\mathscr{R}^\infty_{\mathscr{N}}$ of the results obtained in **[13, 14]** for $\operatorname{alg}\mathscr{N}$.)

If $\{e_k\}_{k=1}^\infty$ is an ONB and $\mathscr{N} = \{\{0\}, [\bigvee_{k=1}^n \{e_k\}]_{n=1}^\infty, \mathscr{H}\}$, then $\mathscr{R}^\infty_{\mathscr{N}}$ is, precisely, the family of all those operators $\mathscr{A}$ in $\mathscr{L}(\mathscr{H})$ which admit a triangular matrix with zeroes in the main diagonal, with respect to the basis. In this case, Corollary 2.5 says that $\hat{\mathscr{R}}^\infty_{\mathscr{N}} = (\text{StrQT})_{-1}$. Thus, $\hat{\mathscr{R}}^\infty_{\mathscr{N}}$ can be thought of as the analog of $(\text{StrQT})_{-1}$ for an arbitrary nest $\mathscr{N}$.

The main tools for the characterization of $\hat{\mathscr{R}}^{\infty}_{\mathscr{N}}$ are, in all cases, Theorem 2.3 and its corollaries. Sample results:

(a) If $\mathscr{N}$ is uncountable, then $\hat{\mathscr{R}}^{\infty}_{\mathscr{N}} = \{T \in \mathscr{L}(\mathscr{H}): 0 \in \sigma_e(T)\hat{}\}$.

(b) If $\mathscr{N}$ is countable and includes a chain $\{\mathscr{M}_k\}_{k \in \mathbf{Z}}$ such that $\bigcap_k \mathscr{M}_k = \{0\}$ and $\bigvee_k \mathscr{M}_k = \mathscr{H}$, then $\hat{\mathscr{R}}^{\infty}_{\mathscr{N}} = \{T \in \mathscr{L}(\mathscr{H}): 0 \in \sigma_e(T)\hat{}$ and $\sigma_e(T)\hat{}$ is connected$\}$.

In all cases (in particular, in (a) and (b)), $\hat{\mathscr{R}}^{\infty}_{\mathscr{N}}$ is a closed subset of $\mathscr{L}(\mathscr{H})$, invariant under similarity and compact perturbations. Indeed, the results in **[24]** include precise information about the cases when arbitrarily small compact perturbations of a given operator T belong to $\{UAU^*: U$ is unitary, $A \in \mathscr{R}^{\infty}_{\mathscr{N}}\}$, for a given nest $\mathscr{N}$.

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